Homework 10 - Material from Chapters 10-11

1. Let \( \mathbb{R}[x] \) denote the group consisting of all polynomials (in the variable \( x \)) with real coefficients, under addition. (So each element is a polynomial, the operation is addition, the identity element is the zero polynomial \( f(x) = 0 \), etc.) Determine whether the function \( \varphi : \mathbb{R}[x] \to \mathbb{R}[x] \) given by \( \varphi(f(x)) = f'(x) \) is a homomorphism or not, and justify your answer. If it is a homomorphism, what is its kernel?

Solution: Yes, \( \varphi \) is a homomorphism: Given any \( f, g \in \mathbb{R}[x] \), we have \( \varphi(f(x) + g(x)) = (f(x) + g(x))' = f'(x) + g'(x) = \varphi(f(x)) + \varphi(g(x)) \) by properties of derivatives.

Since the identity of \( \mathbb{R}[x] \) is just the set of all constant functions.

2. Let \( G, H, K \) be groups. Suppose \( \varphi : G \to H \) is a homomorphism and \( \psi : H \to K \) is a homomorphism. Prove that \( \psi \varphi : G \to K \) is also a homomorphism, and describe how \( \ker \varphi \) and \( \ker \psi \varphi \) are related.

Solution: Let \( a, b \in G \) be arbitrary. Then \( \psi \varphi(ab) = \psi(\varphi(ab)) = \psi(\varphi(a)\varphi(b)) \) since \( \varphi \) is OP. But since \( \psi \) is also OP, that equals \( \psi(\varphi(a))\psi(\varphi(b)) = \psi \varphi(a)\psi \varphi(b) \), so \( \psi \varphi \) is also OP and thus a homomorphism.

Suppose \( x \in \ker \varphi \). Then \( x \in G \) and \( \varphi(x) = e_H \), the identity of \( H \). Since \( \psi \) is a homomorphism, \( \psi(e_H) = e_K \), the identity of \( K \). Thus \( \psi \varphi(x) = \psi(\varphi(x)) = \psi(e_H) = e_K \), so \( x \in \ker \psi \varphi \). This proves that \( \ker \psi \subseteq \ker \psi \varphi \).

(On the other hand, \( \ker \psi \) may contain elements other than \( e_H \) in \( H \), in which case \( \ker \psi \varphi \) contains elements not in \( \ker \varphi \), so the two kernels are not generally equal.)

3. Let \( G \) and \( H \) be groups. Define \( \varphi : G \oplus H \to G \) by \( \varphi(g, h) = g \). Show that \( \varphi \) is a homomorphism, and find its kernel. (This is called a projection mapping.)

Solution: Let \( (a, b) \) and \( (c, d) \) in \( G \oplus H \) be arbitrary. Then \( \varphi((a, b)(c, d)) = \varphi((ac, bd)) = ac \), while \( \varphi((a, b))\varphi((c, d)) = ac \). Since they are equal, \( \varphi \) is a homomorphism.

The kernel is all \( (a, b) \in G \oplus H \) such that \( \varphi((a, b)) = e_G \), which is equivalent to \( a = e_G \).

(b \in H \) can be anything.) So the kernel is the set \( \{e_G\} \oplus H \).

4. Use the first isomorphism theorem to prove that \( \mathbb{R}^*/\langle -1 \rangle \approx \mathbb{R}^+ \).

Solution: There are lots of homomorphisms that work. Here’s one: Define \( \varphi : \mathbb{R}^* \to \mathbb{R}^+ \) by \( \varphi(x) = |x| \) (the absolute value of \( x \), not the order). Then \( \varphi(xy) = |xy| = |x||y| = \varphi(x)\varphi(y) \), so \( \varphi \) is a homomorphism. The kernel of \( \varphi \) is all \( x \in \mathbb{R}^* \) such that \( |x| = 1 \), so the kernel is \( \{\pm 1\} \), also known as \( \langle -1 \rangle \). Also, for any \( b \in \mathbb{R}^+ \) we also have that \( b \in \mathbb{R}^* \) and \( \varphi(b) = |b| = b \) since \( b \) is positive, so \( \varphi \) is onto.

Thus by the first isomorphism theorem, the domain of \( \varphi \) / kernel of \( \varphi \) is isomorphic to image of \( \varphi \), so we have \( \mathbb{R}^*/\langle -1 \rangle \approx \mathbb{R}^+ \).
5. Use the first isomorphism theorem to prove that \( \mathbb{Z}_{24}/\langle 6 \rangle \approx \mathbb{Z}_6 \). Can you generalize the statement?

**Solution:** Define \( \varphi : \mathbb{Z}_{24} \to \mathbb{Z}_6 \) by \( \varphi(n) = n \mod 6 \). Then we have \( \varphi(a + b) = \varphi(a) + \varphi(b) \mod 24 = (a + b) \mod 24 \mod 6 \). Since 6 is a divisor of 24, mod 24 then mod 6 is equivalent to just mod 6. So this equals \((a + b) \mod 6 = a \mod 6 + b \mod 6 \mod 6 = \varphi(a) + \varphi(b)\). Thus \( \varphi \) is a homomorphism. Its kernel is all elements of \( \mathbb{Z}_{24} \) that reduce to 0 mod 6, which is to say all multiples of 6 in \( \mathbb{Z}_{24} \), also known as \( \langle 6 \rangle \). Also, if \( x \in \mathbb{Z}_6 \) then also \( x \in \mathbb{Z}_{24} \) and \( \varphi(x) = x \), so \( \varphi \) is onto. Thus by the first isomorphism theorem, the result follows.

6. If \( \varphi \) is a homomorphism from \( \mathbb{Z}_{40} \) onto a group of order 8, find \( \ker \varphi \).

**Solution:** The first iso. theorem says \( \mathbb{Z}_{40}/\ker \varphi \approx \) group of order 8, so the order of \( \mathbb{Z}_{40}/\ker \varphi \) must be 8. That means \( \frac{|\mathbb{Z}_{40}|}{|\ker \varphi|} = 8 \), so \( |\ker \varphi| = 5 \). There is only one subgroup of order 5 in \( \mathbb{Z}_{40} \), namely \( \langle 8 \rangle \), so that must be the kernel of \( \varphi \).

7. Let \( G, H \) be groups. Prove that \( (G \oplus H)/(G \oplus \{e\}) \approx H \).

**Solution:** Define a function \( \varphi : G \oplus H \to H \) by \( \varphi((g, h)) = h \). Similar to number 3 above, \( \varphi \) is a homomorphism, onto, with kernel \( G \oplus \{e_H\} \). Thus by the first isomorphism theorem, \( (G \oplus H)/G \oplus \{e_H\} \approx H \), as desired.

8. Define a function \( \varphi : \mathbb{Z}_{12} \to \mathbb{Z}_{10} \) by \( \varphi(x) = 3x \). Why isn’t \( \varphi \) a homomorphism?

**Solution:** It’s not operation-preserving for all elements: \( \varphi(6 + 6) = \varphi(0) = 0 \), while \( \varphi(6) + \varphi(6) = 18 + 18 \mod 10 = 36 \mod 10 = 6 \). Since these aren’t equal, \( \varphi \) isn’t OP.

9. Define a function \( \varphi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \) by \( \varphi(a, b) = b - a \).

(a) Show that \( \varphi \) is a homomorphism, and find its kernel.

**Solution:** For arbitrary \((a, b), (c, d) \in \mathbb{Z} \oplus \mathbb{Z}\), we have \( \varphi((a, b) + (c, d)) = \varphi((a + c, b + d)) = (b + d) - (a + c) = b + d - a - c = (b - a) + (d - c) = \varphi((a, b)) + \varphi((c, d)) \), so \( \varphi \) is a homomorphism.

Since the identity of \( \mathbb{Z} \) is 0, the kernel of \( \varphi \) is \( \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} | b - a = 0\} \), which is just the set of all ordered pairs with the same integer in each coordinate, or in other words \( \{(a, a) | a \in \mathbb{Z}\} \).

(b) What familiar group is \( (\mathbb{Z} \oplus \mathbb{Z})/(\ker \varphi) \) isomorphic to? (Use the first isomorphism theorem.)

**Solution:** We already have a homomorphism \( \varphi \), so the 1st isomorphism theorem tells us the domain mod the kernel is isomorphic to the image. So \( (\mathbb{Z} \oplus \mathbb{Z})/\ker \varphi \approx \) the image of \( \varphi \). Now for any \( n \in \mathbb{Z} \), we have \((0, n) \in \mathbb{Z} \oplus \mathbb{Z} \) and \( \varphi((0, n)) = n - 0 = n \), so \( \varphi \) is onto, and thus the image is all of \( \mathbb{Z} \).

Therefore, \( (\mathbb{Z} \oplus \mathbb{Z})/\ker \varphi \approx \mathbb{Z} \).
(c) Describe the set $\varphi^{-1}(7)$ (the “pre-image” of 7, that is, the set of elements that map to 7).

Solution: $\varphi^{-1}(7) = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid b - a = 7\}$, or in other words all ordered pairs in which the second coordinate is 7 more than the first coordinate.

10. What is the smallest positive integer $n$ such that there are 3 or more nonisomorphic Abelian groups of order $n$?

Solution: We use the fundamental theorem of finite abelian groups to note that there is only one isomorphism class of abelian groups of any prime order, and two of any prime-squared order. Also there is only one abelian group of order 6, namely $\mathbb{Z}_2 \oplus \mathbb{Z}_3$. However, if the group has order 8, there are 3 isomorphism classes: $\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Thus $n = 8$ is the smallest such positive integer.

11. How many nonisomorphic Abelian groups of order 300 are there? List them.

Solution: Note that $300 = 2^2 \cdot 3 \cdot 5^2$. So using the fundamental theorem, we have any abelian group of order 300 must be isomorphic to one of the following:

$$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25}$$
$$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$$
$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{25}$$
$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$$

Since none of these groups are isomorphic to each other, there are only 4 mutually non-isomorphic abelian groups of order 300.

12. Suppose $G$ is an Abelian group of order 120 and $G$ has exactly 3 elements of order 2. What direct product of cyclic groups is $G$ isomorphic to?

Solution: Since $G$ is abelian of order 120, we know $G$ is isomorphic to exactly one of $\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$, or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$.

Now the first of those groups only has one element of order 2, namely $(4, 0, 0)$. The second has 3 elements of order 2, namely $(2, 0, 0, 0), (2, 1, 0, 0)$, and $(2, 1, 0, 0)$. The last group has more than 3 elements of order 2, since you can create such an element by using $(*, *, *, 0, 0)$ where each $*$ can be either 0 or 1, and at least one of them must be 1. (This gives 7 possible elements.) Therefore, we must have

$$G \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5.$$
\[
\begin{array}{ccc}
\mathbb{Z}_3 \oplus \mathbb{Z}_5 & \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \\
\hline
\text{element of order 3} & (3, 0) & (1, 0, 0) \\
\text{element of order 5} & (0, 1) & (0, 1) \\
\text{element of order 15} & (3, 1) & (1, 0, 1) \\
\text{element of order 9} & (1, 0) & \text{none} \\
\end{array}
\]

(Since the LCM of the orders of elements of \( \mathbb{Z}_3, \mathbb{Z}_3, \) and \( \mathbb{Z}_5 \) can never be 9, there are no elements of order 9 in \( \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \).)

This proves the required statements.