Canonical Dimension of
Projective Homogeneous Varieties
of Inner Type A and Type B

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

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# Table of Contents

1 Introduction ......................................................... 1  
   1.1 Outline ....................................................... 3  

2 Canonical dimension ............................................... 5  
   2.1 Generic splitting fields ...................................... 5  
   2.2 Rational compressions ....................................... 7  
   2.3 Essential dimension ......................................... 11  
   2.4 Example ......................................................... 12  

3 Projective homogeneous varieties ................................. 14  
   3.1 Semisimple algebraic groups ................................ 14  
   3.2 Projective homogeneous varieties .......................... 15  

4 Reductions and known results ..................................... 17  
   4.1 Projective $\text{PGL}(A)$-homogeneous varieties .......... 17  
   4.2 Projective $\text{O}^+(\varphi)$-homogeneous varieties ...... 21  

5 Higher Witt indices of quadratic forms .......................... 25  
   5.1 Definition ..................................................... 25  
   5.2 The Witt index of $\varphi_{F(X_2)}$ ............................ 26  

6 Motivic decomposition ............................................... 28  
   6.1 Definition ..................................................... 28
6.2 Motivic decomposition ........................................... 29
6.3 Generalized Severi-Brauer varieties \( X_e(A) \) ..................... 30
6.4 Orthogonal Grassmannians \( X_2(\varphi) \) ......................... 31

7 Correspondences and \( p \)-incompressibility ...................... 33
7.1 Criterion for \( p \)-incompressibility ............................... 33
7.2 Even multiplicities .................................................. 34
7.3 Applications via indices ............................................ 36

8 Combinatorial shapes ................................................. 38
8.1 Notation and definitions ............................................ 38
8.2 Multiplication in the Chow rings ................................. 41
8.3 Proof of the lemma .................................................. 43

9 Incompressibility theorems ........................................... 46
9.1 Generalized Severi-Brauer varieties \( X_e(A) \) ..................... 46
9.2 Orthogonal Grassmannians \( X_2(\varphi) \) ......................... 47

References ............................................................... 50
List of Figures

1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 39
2 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 40
3 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 40
4 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 41
5 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 41
6 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 44
LIST OF TABLES

1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 32
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We prove incompressibility for two new infinite families of projective homogeneous varieties.

Let \( e = 2^a \) for \( a \in \mathbb{N} \). For a central division algebra \( A \) with \( \text{ind } A = 2^{a+1} \), we prove that the generalized Severi-Brauer variety \( X_e(A) \) of right ideals of \( A \) of reduced dimension \( e \) is 2-incompressible.

Let \( \varphi \) be a nondegenerate quadratic form on a vector space \( V \) of dimension \( 2n + 1 \) for \( n \in \mathbb{N} \), and define \( X_d(\varphi) \) to be the orthogonal Grassmannian of \( d \)-dimensional totally isotropic subspaces of \( V \) with respect to \( \varphi \). We give a sufficient condition for \( X_2(\varphi) \) to be 2-incompressible, generalizing in a natural way the known sufficient conditions for \( X_1(\varphi) \) and \( X_n(\varphi) \).
CHAPTER 1

Introduction

The idea of canonical dimension has roots in a classic question of algebraic geometry: over which fields does a system of polynomial equations have a solution, or, in geometric language, over which fields does a scheme have a rational point? From this perspective, the canonical dimension of a scheme $X$ over a field $F$ is a nonnegative integer which measures how far the base field $F$ must be extended for $X$ to have a rational point. Indeed, if $X$ is a smooth and geometrically irreducible variety, then the canonical dimension of $X$ is zero if and only if $X$ already has a rational point over $F$ itself.

The field extensions of $F$ over which a scheme $X$ has a rational point are called splitting fields of $X$. The canonical dimension of a scheme was originally defined to be the minimal transcendence degree of a “generic” splitting field. In the special case of regular complete varieties, there is an equivalent geometric definition which is more intuitive: the dimension of the smallest subvariety onto which a variety can be “rationally compressed” (i.e., mapped via a rational morphism). If a variety has a rational point over $F$, then the variety can be compressed onto the point and the canonical dimension is zero. At the opposite extreme, when a variety cannot be compressed at all (hence is “incompressible”), the canonical dimension is equal to the dimension of the entire variety.

Canonical dimension was introduced by Berhuy and Reichstein in 2005 [2], with the involvement of J.-P. Serre, in response to a question originally posed by
Kersten and Rehmann in 1994 [14]. Earlier, generic splitting fields were studied in the 1950’s and 60’s by Amitsur [1] and Roquette [22] for central simple algebras and in the 1970’s by Knebusch [15, 16] for quadratic forms. The broader notion of canonical $p$-dimension was introduced by Karpenko and Merkurjev in 2006 [13]. A connection with essential $p$-dimension via detection functors has recently been elucidated by Merkurjev in his comprehensive exposition [20] of essential dimension.

Current work focuses on the canonical dimension of projective homogeneous varieties. Also known as twisted flag varieties, these are highly-symmetric varieties which are acted upon by semisimple algebraic groups. Prior to the work recorded herein, canonical dimension had been computed for just a few classes of projective homogeneous varieties. It was shown in [2] based on Karpenko’s [11] that the Severi-Brauer variety of a central division algebra of prime-power index is incompressible. Colliot-Thélène, Karpenko, and Merkurjev addressed the index 6 case in [8]. Karpenko and Merkurjev computed the canonical dimension of a quadric in [12] and found a sufficient condition for incompressibility of the maximal orthogonal Grassmannian associated to a quadratic form in [10], building on the work of Vishik [29]. Semenov and Zainoulline recently extended the former result to Hermitian quadrics [25].

This paper proves incompressibility for two new infinite families of projective homogeneous varieties: generalized Severi-Brauer varieties of rank $2^a$ of central division algebras of index $2^{a+1}$, $a \in \mathbb{N}$, and (under appropriate hypotheses) orthogonal Grassmannians of rank 2. Our proofs exploit a criterion for $p$-incompressibility of a variety $X$ involving the vanishing (modulo $p$) of the multiplicities of a correspondence (between $X$ and itself) and its transpose. The advantage of this criterion is that it can be checked on cycles in the images of
the direct summands in a decomposition of the middle-dimensional component of the Chow group of $X \times X$. We obtain such summands by computing motivic decompositions following the prescription of Chernousov and Merkurjev in [6]. By keeping track of the (Schur) indices of central simple algebras and the Witt indices of quadratic forms, we are able to check our criterion for each of the direct summands. Our proof in the orthogonal Grassmannian case also involves a novel application of multiplicity formulas from the work of Pragacz and Ratajski [21] on computation in the relevant Chow rings, involving combinatorial “diagrams” and “shapes.”

Subsequent to the completion of this work, Karpenko generalized one of our results, using a more sophisticated motivic category obtained via idempotent completion. He has computed canonical $p$-dimension ($p$ a prime) for all finite direct products of projective $\text{PGL}(A)$-homogeneous varieties, where $A$ is a central simple algebra over a field $F$. This result has played a key role in his proof that a non-hyperbolic orthogonal involution on a central simple $F$-algebra $A$ remains non-hyperbolic over the function field of the Severi-Brauer variety of $A$.

1.1 Outline

The main incompressibility theorems are proven at the end in chapter 9.

We begin in chapters 2 and 3 with the definitions and basic properties of canonical dimension and projective homogeneous varieties. Chapter 2 describes the three different formulations of canonical dimension, followed by an explicit example of a variety with canonical dimension strictly between zero and the usual dimension.

In chapter 4, we focus our attention on the projective homogeneous varieties
of semisimple algebraic groups of inner type A and of type B. After obtaining some initial reductions of the general problem of computing canonical dimension of these varieties, we place our present work in the context of the cases which are already understood.

Chapter 5 recalls the definition of the higher Witt indices of a quadratic form and proves two results which will be needed in the final chapter.

In chapter 6, we define the category of graded correspondences. After describing the Chernousov-Merkurjev motivic decomposition, we compute it for the two cases with which we are concerned, namely certain generalized Severi-Brauer varieties and the rank-2 orthogonal Grassmannians.

Chapter 7 proves the criterion we will use for $p$-incompressibility and then relates it to the degrees of the closed points of certain varieties. This result is then applied to obtain two propositions which will take care of most of the direct summands in our proofs in chapter 9.

Chapter 8 provides the one remaining (rather technical) missing piece which will enable us to deal with the most difficult summand in the incompressibility proof for orthogonal Grassmannians of rank 2. The argument is based on Pieri-type formulas developed by Pragacz and Ratajski for multiplication in certain Chow rings. While such formulas still exist for orthogonal Grassmannians of higher rank, the combinatorics no longer work as desired, which is the reason that our proof does not generalize. The fact that the combinatorics do work in the rank 2 case feels a bit miraculous.

Finally, in chapter 9 we put together all that has come before to prove 2-incompressibility for our two infinite families of projective homogeneous varieties.
CHAPTER 2

Canonical dimension

In this paper, a scheme over a field $F$ means a separated scheme of finite type over $F$, and a variety over $F$ is an integral scheme over $F$.

We begin by stating the original definition of canonical $p$-dimension of a scheme $X$ over a field $F$, in terms of generic splitting fields. When $X$ is a regular complete variety, this is equivalent to a second definition involving rational compressions, and (assuming $X$ is smooth) to a third definition which exhibits canonical $p$-dimension as a special case of essential $p$-dimension. We prove the equivalence of the first two definitions for regular complete varieties. For a comprehensive exposition of canonical $p$-dimension from the perspective of essential $p$-dimension, we refer the reader to Merkurjev’s [20].

In the last section of this chapter, we give an example of a smooth complete variety $X$ over the real numbers with dimension 2 and canonical dimension 1, together with an explicit rational compression which realizes the canonical dimension.

2.1 Generic splitting fields

In all that follows, let $p$ be a prime or 0. When we call an integer prime to 0, we mean that it equals 1.
The canonical $p$-dimension of a scheme $X$ over $F$ was originally defined [2, 13] in terms of generic splitting fields, which in turn are defined using valuation rings and places.

Let $K$ be a field. A subring $R \subset K$ is a valuation ring of $K$ if, for every nonzero $x \in K$, at least one of $x, x^{-1}$ is in $R$.

**Proposition 2.1.1.** A valuation ring $R$ of a field $K$ is a local domain.

**Proof.** Any subring of a field is a domain. To prove that $R$ is local, it suffices to show that the nonunits form an ideal. Suppose $x, y \in R$ are not units. It follows immediately that for $r \in R$, $rx$ is not a unit. Without loss of generality, assume $x/y \in R$. Then $1 + x/y = (y + x)/y \in R$, which implies that $y + x$ is not a unit. \hfill $\square$

For fields $K$ and $L$, a place $\pi : K \rightarrow L$ is a (local) ring homomorphism $f : R \rightarrow L$, for some valuation ring $R$ of $K$. The place $\pi$ is said to be defined on $R$.

A place $K \rightarrow L$ between two extensions of a field $F$ is called an $F$-place if it is defined and equal to the identity on $F$.

Two places $K \rightarrow L$ and $L \rightarrow M$ can be composed. Suppose they are given by ring homomorphisms $f : R \rightarrow L$ and $g : S \rightarrow M$. Then $f^{-1}(S) \subset R$ is a valuation ring because, for any nonzero $x \in R$, either $f(x)$ or $f(x^{-1}) = f(x)^{-1}$ must be in $S$. We define the composition place $K \rightarrow M$ to be the composition of ring homomorphisms $f^{-1}(S) \rightarrow S \rightarrow M$.

Let $X$ be a scheme over a field $F$, and let $p$ be a prime or zero. For a field extension $K$ of $F$, we write $X(K)$ for the set of $K$-points of $X$, i.e. morphisms $\text{Spec } K \rightarrow X$ over $\text{Spec } F$. The field $K$ is called a splitting field of $X$ (or is said to split $X$) if $X(K) \neq \emptyset$. We will also say that $X$ splits over $K$. A splitting field
$K$ is called $p$-generic if, for any splitting field $L$ of $X$, there is an $F$-place $K \rightarrow L'$ for some finite extension $L'/L$ of degree prime to $p$. In particular, $K$ is 0-generic if for any splitting field $L$ there is an $F$-place $K \rightarrow L$.

We are now ready to state the original definition of canonical $p$-dimension [2, 13].

**Definition 2.1.2.** The canonical $p$-dimension of a scheme $X$ over $F$ is the minimal transcendence degree of a $p$-generic splitting field $K$ of $X$. If no such field exists, we say the canonical $p$-dimension is $\infty$.

### 2.2 Rational compressions

For regular complete varieties, canonical $p$-dimension has an equivalent definition which is more geometric in nature [13].

**Proposition 2.2.1** (Karpenko, Merkurjev). Let $X$ be a regular complete variety over $F$. Then $\text{cdim}_p(X)$ is the minimal dimension of the image of a morphism $X' \rightarrow X$, where $X'$ is a variety over $F$ admitting a dominant morphism $X' \rightarrow X$ with $F(X')/F(X)$ of degree prime to $p$. The canonical 0-dimension of $X$ is thus the minimal dimension of the image of a rational morphism $X \dashrightarrow X$.

In the case $p = 0$, we will drop the $p$ and speak simply of generic splitting fields and canonical dimension $\text{cdim}(X)$. Intuitively, the canonical dimension of $X$ is the dimension of the smallest closed subvariety $Y \subset X$ onto which $X$ can be “compressed” by a rational morphism $X \dashrightarrow Y$.

Before proving proposition 2.2.1, we establish two lemmas.

**Lemma 2.2.2.** Let $X$ be a variety over a field $F$, with $E$ an extension field of $F$. If $X$ is complete and there exists an $F$-place $F(X) \rightarrow E$, then $X(E) \neq \emptyset$. If
$X$ is regular and $X(E) \neq \emptyset$, then there exists a geometric $F$-place $F(X) \twoheadrightarrow E$.

Proof. Suppose $X$ is complete and we are given an $F$-place $\pi : F(X) \rightarrow E$ defined on the valuation ring $R$ with maximal ideal $M$. Since $E$ is a field, the ring homomorphism $R \rightarrow E$ factors through $R/M$. By the valuative criterion of properness, there exists a unique morphism $f$ making the following diagram commutative:

$$
\begin{array}{ccc}
\text{Spec } F(X) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec } R & \longrightarrow & \text{Spec } F
\end{array}
$$

Thus the image $x \in X$ under $f$ of the closed point of $\text{Spec } R$ is the unique point of $X$ such that $R$ dominates the local ring $O_{x,X}$. Composing the induced homomorphism of fields $F(x) \rightarrow R/M$ with $R/M \rightarrow E$ from above yields a morphism $\text{Spec } E \rightarrow X$ of schemes, which is said to be induced by the place $\pi$.

For $X$ regular, suppose there is a morphism $\text{Spec } E \rightarrow X$ with image $\{x\} \in X$ and with corresponding inclusion of fields $F(x) \hookrightarrow E$. Let $a_1, a_2, \ldots, a_n$ be a regular system of parameters in $R := O_{x,X}$. For $i = 1, \ldots, n$, define $M_i$ to be the ideal of $R$ generated by $a_1, a_2, \ldots, a_i$, $R_i := R/M_i$, and let $F_i$ be the quotient field of $R_i$. For $i = 1, \ldots, n-1$, define $P_i := M_{i+1}/M_i$. Then the $(R_i)_{P_i}$ are noetherian regular local rings of dimension one, namely discrete valuation rings. Since $(R_i)_{P_i}$ has quotient field $F_i$ and residue field $F_{i+1}$, it defines an $F$-place $\pi_i : F_i \rightarrow F_{i+1}$. We can compose

$$F(X) = F_0 \rightarrow F_1 \rightarrow \cdots \rightarrow F_n = F(x) \hookrightarrow E$$

to obtain the desired geometric $F$-place $F(X) \twoheadrightarrow E$.

Lemma 2.2.3. If $X$ is regular, then $F(X)$ is a generic splitting field of $X$. 

8
Proof. Let $E/F$ be a splitting field of $X$, so that $X(E) \neq \emptyset$. By the previous lemma, there is an $F$-place $F(X) \rightarrow E$.  

It follows immediately from this lemma that $\text{cdim}_p(X) \leq \dim(X)$ for regular $X$.

Proof of proposition 2.2.1. Take $X$ to be a regular variety. Let $Y \subset X$ be the closure of the image of a morphism $X' \rightarrow X$, for $X'$ a variety over $F$ admitting a dominant morphism $X' \rightarrow X$ with $F(X')/F(X)$ of degree prime to $p$. There is an induced dominant morphism $X' \rightarrow Y$, hence $F(Y)$ injects into $F(X')$. For any splitting field $L$ of $X$, there exists a geometric place $F(X) \rightarrow L$, since $F(X)$ is generic. By [13, Lemma 3.2], there exists a $p$-coprime extension $L'$ of $L$ with a place $F(X') \rightarrow L'$, as illustrated below. Composing with $F(Y) \hookrightarrow F(X')$, we get a place $F(Y) \rightarrow L'$, which shows that $F(Y)$ is $p$-generic. Thus $\text{cdim}_p(X) \leq \text{trdeg}(Y) = \dim(Y)$, proving the first half of the proposition.

Now let $X$ be complete. Take $E$ to be a $p$-generic splitting field of $X$ of minimal transcendence degree. Since $E$ is a splitting field, there exists a morphism $\text{Spec} E \rightarrow X$. Setting $Y \subset X$ to be the closure of the image of this morphism, we have $F(Y) \hookrightarrow E$ and $\dim(Y) = \text{trdeg}(F(Y)) \leq \text{trdeg}(E) = \text{cdim}_p(X)$. Since $E$ is $p$-generic, $F(Y)$ is also. By definition, there is a $p$-coprime extension $L'$ of $F(X)$ with a place $F(Y) \rightarrow L'$. Let $X'$ be a variety with $F(X') \simeq L'$. We obtain rational morphisms $\varphi : X' \rightarrow X$ (with $F(X')/F(X)$ of degree prime to $p$) and $X' \rightarrow Y$ (by Lemma 2.2.2, using that $Y$ is complete). Composing the latter with $Y \rightarrow X$, we get $\psi : X' \rightarrow X$, with image of dimension less than or equal to $\dim(Y) \leq \text{cdim}_p(X)$. Replacing $X'$ by the (nonempty) intersection of
the domains of definition of $\varphi$ and $\psi$, the rational morphisms become morphisms, and we are done.

Recall that a point $x$ of a scheme $X$ over the field $F$ is called rational if its residue field $F(x) = F$.

**Proposition 2.2.4.** If a scheme $X$ over the field $F$ has a rational point, then $\text{cdim}(X) = 0$. If $X$ is a smooth and geometrically irreducible variety, then the converse holds as well.

**Proof.** If the scheme $X$ has a rational point over $F$, then $F$ is a generic splitting field, so $\text{cdim}(X) = \text{trdeg}(F) = 0$.

Now assume $X$ is smooth and geometrically irreducible and suppose $\text{cdim}(X) = 0$. Then there exists a subvariety $Y \subset X$ of dimension 0 and a (dominant) rational map $X \to Y$. This induces field inclusions $F \subset F(Y) \subset F(X)$ with $F(Y)$ a separable extension of $F$, by the smoothness assumption [27, Prop. 7.103]. Since $X$ is geometrically irreducible, every separable element of $F(X)$ lies in $F$, so $F(Y) = F$ and $Y$ consists of a rational point of $X$.

If we remove either the smooth or the geometrically irreducible assumption, the converse no longer holds. For example, if $K$ is a purely inseparable extension of $F$, then $\text{Spec}(K)$ is geometrically irreducible but not smooth. If instead we take $K$ to be a separable extension of $F$, then $\text{Spec}(K)$ is smooth, but not geometrically irreducible. In each case, the canonical dimension is clearly 0, yet the variety has no rational point.

**Definition 2.2.5.** When a smooth complete variety $X$ has canonical $p$-dimension as large as possible, namely $\text{cdim}_p(X) = \text{dim}(X)$, we say that $X$ is $p$-incompressible.

10
It follows immediately that if $X$ is $p$-incompressible, it is also *incompressible* (i.e. 0-incompressible).

### 2.3 Essential dimension

A third equivalent definition of canonical $p$-dimension, for smooth complete varieties, is in terms of essential $p$-dimension [20].

We write $\text{Fields}/F$ for the category of finitely generated field extensions of $F$ and field homomorphisms over $F$.

Let $T : \text{Fields}/F \to \text{Sets}$ be a functor. Given a morphism $f : L \to L'$ in $\text{Fields}/F$ and an element $\alpha \in T(L)$, we write $\alpha_{L'}$ for the image of $\alpha$ under $T(f)$.

Let $K, L \in \text{Fields}/F$ and take elements $\beta \in T(K), \alpha \in T(L)$. Then we write $\alpha \succ_p \beta$ if there exists a finite field extension $L'/L$ of degree prime to $p$ and a morphism $K \to L'$ such that $\alpha_{L'} = \beta_{L'}$. If $p = 0$, we drop the $p$ and simply write $\alpha \succ \beta$ when there is a morphism $K \to L$ with $\alpha = \beta_L$.

**Definition 2.3.1.** The *essential $p$-dimension of $\alpha_*$*, denoted $\text{ed}_p(\alpha)$, is the minimal transcendence degree of a field $K \in \text{Fields}/F$ such that there exists an element $\beta \in T(K)$ with $\alpha \succ_p \beta$. The *essential $p$-dimension of the functor $T$* is the integer

$$\text{ed}_p(T) := \max\{\text{ed}_p(\alpha)\}$$

where the maximum is taken over all $L \in \text{Fields}/F$ and $\alpha \in T(L)$. When $p = 0$, we drop the $p$ and write $\text{ed}(T)$ for the *essential dimension of $T$*.

Any scheme $X$ over a field $F$ can be viewed as a functor “of points” $X : \text{Fields}/F \to \text{Sets}$ which maps $K \in \text{Fields}/F$ to the set of $K$-points of $X$, namely $X(K) = \text{Mor}_F(\text{Spec}K, X)$. In this way, we can speak of the essential
p-dimension of a scheme $X$ over $F$, which in fact always equals $\dim(X)$ [20, Prop. 1.2].

To obtain a more interesting invariant for schemes, we must modify the functor. Given any functor $T : \text{Fields}/F \to \text{Sets}$, define its detection functor $	ilde{T} : \text{Fields}/F \to \text{Sets}$ by setting

$$
\tilde{T}(K) := \begin{cases} 
\{K\} & \text{if } T(K) \neq \emptyset, \\
\emptyset & \text{if } T(K) = \emptyset.
\end{cases}
$$

**Proposition 2.3.2** (Merkurjev). For a smooth complete variety $X$ viewed as a functor from $\text{Fields}/F \to \text{Sets}$, the essential p-dimension of the detection functor $\tilde{X}$ is equal to $\cdim_p(X)$.

### 2.4 Example

We now give a concrete example of a smooth complete variety $X$ over the real numbers with $0 < \cdim X < \dim X$. Let $X$ be the projective quadric surface associated to the quadratic Pfister form

$$
\langle\langle -1, -1 \rangle\rangle = \langle 1, 1 \rangle \otimes \langle 1, 1 \rangle = \langle 1, 1, 1, 1 \rangle.
$$

This form is clearly anisotropic over the real numbers. Since a Pfister form becomes hyperbolic over its function field, we have $i_1(X) = 2$ and

$$
\cdim X = \dim X + 1 - i_1(X) = 1,
$$

by [10, Th. 90.2].

We know from the definition of canonical dimension in terms of rational compressions that there must be a subvariety $Y$ of $X$ with $\dim Y = 1$ and a rational morphism $X \dashrightarrow Y$. If we write

$$
X = \text{Proj } \mathbb{R}[x, y, z, w]/(x^2 + y^2 + z^2 + w^2),
$$

12
then we can consider the subvariety

\[ Y = \text{Proj } \mathbb{R}[x, y, z, w]/(x^2 + y^2 + z^2 + w^2, w) \]
\[ \cong \text{Proj } \mathbb{R}[r, s, t]/(r^2 + s^2 + t^2). \]

A rational morphism \( X \rightarrow Y \) is then given by

\[ r \mapsto xz + yw \]
\[ s \mapsto yz - xw \]
\[ t \mapsto x^2 + y^2. \]

This is well defined because

\[ r^2 + s^2 + t^2 \mapsto (xz + yw)^2 + (yz - xw)^2 + (x^2 + y^2)^2 = (x^2 + y^2)(x^2 + y^2 + z^2 + w^2). \]
CHAPTER 3

Projective homogeneous varieties

The varieties we would like to show to be 2-incompressible are all projective homogeneous varieties of semisimple algebraic groups. We here introduce the concepts and notation which will be needed in the subsequent chapters. For further details, we refer the reader to [3] and [17, Chapter VI].

3.1 Semisimple algebraic groups

Throughout this paper, $G$ will be a semisimple algebraic group over a field $F$. A torus [17, p. 333] is split if it is isomorphic to a product $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$ of copies of the multiplicative group. We say that the group $G$ is split if it has a split maximal torus.

Every group $G$ over $F$ is split over the separable closure $F_{\text{sep}}$ of $F$. Two semisimple algebraic groups $G$ and $G'$ over $F$ are called twisted forms of each other if they are isomorphic over $F_{\text{sep}}$. Every semisimple algebraic group is a twisted form of some split group.

**Definition 3.1.1.** A parabolic subgroup $P$ of a semisimple algebraic group $G$ is a closed subgroup of $G$ such that the factor group $G/P$ is a projective variety.

Assume for the next two paragraphs that $G$ is split. We can fix a split maximal torus $T$, a root system $\Sigma$ with respect to $T$, a Borel subgroup $B \supset T$, and a set
of simple roots $\Pi \subset \Sigma$ corresponding to $B$, along with a set of positive roots $\Sigma^+ \subset \Sigma$. Let $W$ be the Weyl group of $\Sigma$.

A closed subgroup $P$ of $G$ is parabolic if and only if it contains a Borel subgroup [3, p. 148]. We say a parabolic subgroup is standard if it contains the $B$ we have fixed above. All Borel subgroups are conjugate to one another, hence every parabolic subgroup is conjugate to a standard one. Standard parabolic subgroups are in one-to-one correspondence with subsets $S \subset \Pi$. Let the standard parabolic subgroup corresponding to a subset $S$ be denoted $P_S$. A parabolic subgroup conjugate to $P_S$ is said to be of type $S$.

If $G$ is not split, we can still introduce the notation above for the split $(G)_{F_{\text{sep}}}$. A parabolic subgroup $P$ of $G$ is then said to be of type $S$ if it is of type $S$ over $F_{\text{sep}}$.

The projective homogeneous varieties with which we are concerned all correspond to one of the semisimple algebraic groups $\text{PGL}(A)$, for a central division algebra $A$, or $\text{O}^+(\varphi)$, for a nondegenerate quadratic form $\varphi$ on a vector space of odd dimension (see [17] for definitions). These groups are of “inner type,” and to restrict attention to groups of this type will enable a simpler statement of the motivic decomposition in chapter 6.

**Definition 3.1.2.** A semisimple algebraic group $G$ is of inner type if the $*$-action (see [26]) of the Galois group $\text{Gal}(F_{\text{sep}}/F)$ on the Dynkin diagram of $F$ is trivial.

### 3.2 Projective homogeneous varieties

We now give the definition of a projective homogeneous variety of a semisimple algebraic group $G$.

**Definition 3.2.1.** For a semisimple algebraic group $G$, a projective homoge-
neous $G$-variety is a variety $X$ such that $(X)_{F_{\text{sep}}}^{\text{F}}$ is isomorphic to a factor group $(G)_{F_{\text{sep}}}^{\text{F}}/P$ for some parabolic subgroup $P$ of $(G)_{F_{\text{sep}}}^{\text{F}}$. If $P$ is of type $S$, then we can equivalently say that $X$ is (isomorphic to) the variety of parabolic subgroups of $G$ of type $S$. This projective homogeneous $G$-variety of type $S$, which we denote by $H_S$, is defined over $F$ if and only if $S$ is invariant under the $\ast$-action. Thus, if $G$ is of inner type, then $H_S$ is defined over $F$ for all subsets $S \subset \Pi$.

For example, $H_{\emptyset}$ is the variety of all Borel subgroups of $G$, while $H_{\Pi} = \text{Spec } F$, since the only subgroup of type $\Pi$ is $G$ itself.
CHAPTER 4

Reductions and known results

When two schemes \( X \) and \( Y \) over a field \( F \) have the same class of splitting fields, we call them equivalent and write \( X \sim Y \). In this case

\[
\text{cdim}_p(X) = \text{cdim}_p(Y)
\]

for all \( p \).

We now consider the general problem of computing canonical \( p \)-dimension of projective \( G \)-homogeneous varieties, where \( G \) is an algebraic group of one of the forms \( \text{PGL}(A) \) or \( \text{O}^+(\varphi) \) mentioned above. For each of the two types of groups, we use the notion of equivalence to reduce the general problem to certain essential cases, and we then review the results which are known to date.

4.1 Projective \( \text{PGL}(A) \)-homogeneous varieties

Let \( A \) be a central division algebra over a field \( F \) with \( \text{ind} A = n \). The projective homogeneous varieties associated to the group \( \text{PGL}(A) \) can be described as follows.

For integers \( 1 \leq d_1 < d_2 < \cdots < d_k \leq n - 1 \), define \( X_{d_1,d_2,\ldots,d_k}(A) \) to be the variety of flags of right ideals \( I_1 \subset I_2 \subset \cdots \subset I_k \) of \( A \) with \( I_i \) of reduced dimension \( d_i \). When the algebra \( A \) is understood, we write simply \( X_{d_1,d_2,\ldots,d_k} \).
If \(e_1, \ldots, e_n\) are the standard basis vectors of \(\mathbb{R}^n\), we may take
\[
\Pi = \{\alpha_1 := e_1 - e_2, \ldots, \alpha_{n-1} := e_{n-1} - e_n\}
\]
to be a set of simple roots for the root system \(\Sigma\) associated to \(\text{PGL}(A)\). Then \(X_{d_1, d_2, \ldots, d_k}\) is just the projective \(\text{PGL}(A)\)-homogeneous variety of type
\[
S = \Pi \setminus \{\alpha_{d_1}, \alpha_{d_2}, \ldots, \alpha_{d_k}\}.
\]

When \(k = 1\) we get the generalized Severi-Brauer varieties \(X_{d_1}(A)\) of \(A\). In particular, \(X_1(A)\) is the Severi-Brauer variety of \(A\).

It is known [17, Th. 1.17] that the generalized Severi-Brauer variety \(X_{d_1}(A)\) has a rational point over an extension field \(K/F\) if and only if the index \(\text{ind} A_K\) divides \(d_1\). As a consequence, \(X_{d_1}(A) \sim X_d(A)\), where \(d := \gcd(\text{ind} A, A_1)\). We record the easy generalization of this fact to varieties \(X_{d_1, d_2, \ldots, d_k}(A)\).

**Proposition 4.1.1.** If \(d := \gcd(\text{ind} A, d_1, d_2, \ldots, d_k)\), then
\[
X_{d_1, d_2, \ldots, d_k}(A) \sim X_d(A)
\]
and thus, for any \(p\),
\[
\text{cdim}_p(X_{d_1, d_2, \ldots, d_k}(A)) = \text{cdim}_p(X_d(A)).
\]

**Proof.** If \(X_{d_1, d_2, \ldots, d_k}(A)\) has a rational point over an extension field \(K/F\), then by definition \(A_K\) has right ideals of reduced dimensions \(d_1, d_2, \ldots, d_k\). This is the case if and only if \(\text{ind} A_K\) divides each of the \(d_i\), or equivalently, \(\text{ind} A_K\) divides \(d\) (since \(\text{ind} A_K\) always divides \(\text{ind} A\)).

Reading the argument backwards, \(\text{ind} A_K\) dividing \(d\) implies the existence of right ideals \(I_1, I_2, \ldots, I_k\) in \(A_K\) with reduced dimensions \(d_1, d_2, \ldots, d_k\). In fact, the \(I_1, I_2, \ldots, I_k\) can be chosen to form a flag. Suppose \(d_i = m_i \text{ ind} A_K\) and \(A_K \simeq M_{t(D)}\)
for some division algebra $D$. Then we take $I_i$ to be the set of matrices in $M_t(D)$ whose $t - m_i$ last rows are zero.

Hence it is enough to compute $\text{cdim}_p(X_d(A))$ for $d$ dividing $\text{ind} A$.

If the index of $A$ factors as $\text{ind} A = q_1 q_2 \cdots q_r$ with the $q_j$ powers of distinct primes $p_j$, then there exist central division algebras $A_j$ of index $q_j$ for $j = 1, \ldots, r$ such that 

$$A \simeq A_1 \otimes A_2 \otimes \cdots \otimes A_r.$$ 

**Proposition 4.1.2.** Given a positive integer $1 \leq d \leq \text{ind} A - 1$, with $q_j$ as above, define $e_j := \gcd(d, q_j)$ for $j = 1, \ldots, r$. Then 

$$X_d(A) \sim X_{e_1}(A_1) \times X_{e_2}(A_2) \times \cdots \times X_{e_r}(A_r)$$

and thus, for any $p$,

$$\text{cdim}_p(X_d(A)) = \text{cdim}_p(X_{e_1}(A_1) \times X_{e_2}(A_2) \times \cdots \times X_{e_r}(A_r)).$$

**Proof.** The variety $X_d(A)$ has a rational point over an extension field $K/F$ if and only if $\text{ind} A_K$ divides $d$. Because 

$$\text{ind} A_K = (\text{ind}(A_1)_K) \cdots (\text{ind}(A_r)_K),$$

this condition is equivalent to $\text{ind}(A_j)_K$ dividing $d$ for all $j$, or to $\text{ind}(A_j)_K$ dividing $e_j$ for all $j$ (since $\text{ind}(A_j)_K$ always divides $\text{ind} A_j = q_j$). This holds if and only if each $X_{e_j}(A_j)$ has a rational point over $K$, which is equivalent to the product of the $X_{e_j}(A_j)$ having a rational point over $K$. 

The proposition gives the following upper bound on canonical $p$-dimension:

$$\text{cdim}_p(X_d(A)) \leq \dim \prod_{j=1}^r X_{e_j}(A_j) = \sum_{j=1}^r \dim X_{e_j}(A_j) = \sum_{j=1}^r e_j(q_j - e_j). \quad (4.1)$$
If $p$ is prime, then there exists a finite, $p$-coprime extension $K$ of $F$ which splits the algebras $A_j$ for all $j$ with $p_j \neq p$. Since canonical $p$-dimension does not change under such an extension [20, Prop. 1.5 (2)], $\text{cdim}_p(X_d(A)) = 0$ unless some $p_s = p$, in which case

$$\text{cdim}_p(X_d(A)) = \text{cdim}_p(X_{e_s}(A_s)).$$

We see that it is enough, when $p$ is prime, to compute the canonical $p$-dimension of varieties of the form $X_{e_s}(A)$ with $\text{ind} \ A$ a prime power divisible by $e$. When $p = 0$, it is enough to compute the canonical dimension of products of such varieties.

We now recall what is already known about the canonical $p$-dimension of Severi-Brauer varieties $X_1(A)$.

By (4.1), we have

$$\text{cdim}_p(X_1(A)) \leq \sum_{j=1}^{r} (q_j - 1). \tag{4.2}$$

If $\text{ind} \ A$ is a power of a prime $p$ (i.e. $r = 1$), it is shown in [2, Th. 11.4], based on Karpenko’s [11, Th. 2.1], that the inequality (4.2) is actually an equality. Thus

$$\text{cdim}_p(X_1(A)) = (\text{ind} \ A) - 1 = \dim X_1(A),$$

which means that $X_1(A)$ is $p$-incompressible (and hence incompressible).

For $p = 0$, (4.2) has been shown also to be an equality in the case $\text{ind} \ A = 6$, provided that $\text{char} \ F = 0$ [8, Th. 1.3]. The authors of [8] suggest that equality may indeed hold for any $X_1(A)$ when $p = 0$.

If $A$ is a central division algebra with $\text{ind} \ A = 4$, the variety $X_2(A)$ is known to be 2-incompressible. Indeed, if the exponent of $A$ is 2, then $X_2(A)$ is isomorphic to a 4-dimensional projective quadric hypersurface called the *Albert quadric* of
A [19, §5.2]. Such a quadric has first Witt index 1 [28, p. 93], hence is 2-incompressible by [10, Th. 90.2]. If the exponent of $A$ is 4, we can reduce to the exponent 2 case by extending to the function field of the Severi-Brauer variety of $A \otimes A$.

In this paper, we show 2-incompressibility for an infinite family of varieties which includes the varieties of the form $X_2(A)$ (with ind $A = 4$) mentioned above.

**Theorem 4.1.3.** Let $e = 2^a$, $a \in \mathbb{N}$. For a central division algebra $A$ with ind $A = 2^{a+1}$, the variety $X_e := X_e(A)$ is 2-incompressible. Thus

$$\text{cdim}_2(X_e) = \text{cdim}(X_e) = \dim(X_e) = e(2e - e) = e^2 = 4^a.$$  

### 4.2 Projective $O^+(\varphi)$-homogeneous varieties

The *Witt index* $i_0(\varphi)$ of a quadratic form $\varphi$ on a vector space $V$ over a field $F$ is the number of copies of the hyperbolic plane $\mathbb{H}$ which appear in the Witt decomposition of $\varphi$. A quadratic form $\varphi$ is *split* if $i_0(\varphi) = [(\dim \varphi)/2]$, the greatest possible value.

A subspace $W \subset V$ is *totally isotropic* with respect to $\varphi$ if $\varphi(W) = 0$.

Fix a nondegenerate quadratic form $\varphi$ on a vector space $V$ of dimension $2n + 1$, over a field $F$. The projective homogeneous varieties associated to the group $O^+(\varphi)$ can be described as follows.

For integers $1 \leq d_1 < d_2 < \cdots < d_k \leq n$, define $X_{d_1,d_2,\ldots,d_k}(\varphi)$ to be the variety of flags of totally isotropic subspaces $W_1 \subset W_2 \subset \cdots \subset W_k$ of $V$ (with respect to $\varphi$), with $W_i$ of dimension $d_i$. When $V$ and $\varphi$ are understood, we write simply $X_{d_1,d_2,\ldots,d_k}$. 

21
If $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$, we may take

$$\Pi = \{\alpha_1 := e_1 - e_2, \ldots, \alpha_{n-1} := e_{n-1} - e_n, \alpha_n := e_n\}$$

to be a set of simple roots for the root system $\Sigma$ associated to $O^+(\varphi)$. Then $X_{d_1, d_2, \ldots, d_k}$ is just the projective $O^+(\varphi)$-homogeneous variety of type

$$S = \Pi \setminus \{\alpha_{d_1}, \alpha_{d_2}, \ldots, \alpha_{d_k}\}.$$ 

Note that any quadratic form $\varphi$ splits over some field extension of $F$ of degree a power of 2, and thus so does $X_{d_1, d_2, \ldots, d_k}(\varphi)$, which implies $\text{cdim}_p(X_{d_1, d_2, \ldots, d_k}(\varphi)) = 0$ when $p$ is an odd prime. Even for $p \in \{0, 2\}$, it is enough to consider the orthogonal Grassmannians $X_d(\varphi)$, as the next proposition shows.

**Proposition 4.2.1.** If we set $d := \max(d_1, d_2, \ldots, d_k) = d_k$, then

$$X_{d_1, d_2, \ldots, d_k}(\varphi) \sim X_d(\varphi)$$

and thus, for any $p$,

$$\text{cdim}_p(X_{d_1, d_2, \ldots, d_k}(\varphi)) = \text{cdim}_p(X_d(\varphi)).$$

**Proof.** There exists a totally isotropic flag $W_1 \subset W_2 \subset \cdots \subset W_k$ of subspaces with respective dimensions $d_1, d_2, \ldots, d_k$ if and only if there exists a totally isotropic subspace $W$ of degree $d = d_k$. \qed

So we fix $V$ and $\varphi$ and restrict attention to the varieties $X_1, X_2, \ldots, X_n$, where $X_d$ is the variety of $d$-dimensional totally isotropic subspaces of $V$. The variety $X_1$ is simply the projective quadric hypersurface associated to the quadratic form $\varphi$.
We recall what is already known regarding the $X_d$. The following result is proved in [12] and also in [10, Ch. XIV and §90]. See chapter 5 below for the definition of the higher Witt indices $i_1(\varphi), i_2(\varphi), \ldots, i_n(\varphi)$.

**Theorem 4.2.2** (Karpenko, Merkurjev). *If the quadric $X_1$ is anisotropic, then*

$$\text{cdim}_2(X_1) = \text{cdim}(X_1) = \dim(X_1) - i_1(\varphi) + 1.$$  

*In particular, $X_1$ is 2-incompressible if and only if $i_1(\varphi) = 1$.*

At the other extreme is the variety $X_n$ of maximal totally isotropic subspaces of $V$. In [10, Ch. XVI], building on a result of Vishik from [29], the canonical 2-dimension of $X_n$ is computed in terms of the $J$-invariant of $\varphi$. The following result is a corollary.

**Theorem 4.2.3** (Karpenko, Merkurjev). *If $\deg \text{CH}(X_n) = 2^n \mathbb{Z}$, then $X_n$ is 2-incompressible.*

To compute the canonical 2-dimension of a general $X_d$ appears to be difficult because of the complexity of the Chow ring when $d \notin \{1, n\}$. In this paper, we complete a small piece of this general program by determining a sufficient condition for the variety $X_2$ to be 2-incompressible. We assume everywhere that $n \geq 3$, the $n = 2$ case having already been dealt with.

**Theorem 4.2.4.** *If $\deg \text{CH}(X_2) = 4\mathbb{Z}$ and $i_2(\varphi) = 1$, then $X_2$ is 2-incompressible. In particular,*

$$\text{cdim}_2(X_2) = \text{cdim}(X_2) = \dim(X_2) = 4n - 5.$$  

This result concerning $X_2$ is a natural generalization of what is already known about $X_1$ and $X_n$. To see this, note that $X_1$ being anisotropic means that
deg CH($X_1$) = 2$\mathbb{Z}$. Furthermore, deg CH($X_n$) = $2^n\mathbb{Z}$ implies that $j_{n-1}(\varphi) = n-1$, from which it immediately follows that $i_n(\varphi) = 1$. One might then conjecture, for general $d$, that

$$\deg \text{CH}(X_d) = 2^d\mathbb{Z}, \quad i_d(\varphi) = 1$$

are sufficient conditions for $X_d$ to be 2-incompressible.
CHAPTER 5

Higher Witt indices of quadratic forms

Our second main theorem 4.2.4 requires the hypotheses $\deg CH(X_2) = 4\mathbb{Z}$ and $i_2(\varphi) = 1$ on the quadratic form $\varphi$. In this chapter, we conclude from these hypotheses that the usual Witt index $i_0(\varphi_{F(X_2)}) = 2$, which will enable us to eventually make use of Proposition 7.3.2.

5.1 Definition

We begin by recalling the definitions of absolute and relative higher Witt indices, introduced by Knebusch in [15]. Our discussion follows [10, §90]. Let $\varphi$ be a nondegenerate quadratic form over a field $F$ and set $F_0 := F$ and $\varphi_0 := \varphi_{an}$, the anisotropic part of $\varphi$. We proceed to recursively define $F_k := F_{k-1}(\varphi_{k-1})$, $\varphi_k := (\varphi_{F_k})_{an}$ for $k = 1, 2, \ldots$, stopping at $F_h$ such that $\dim \varphi_h \leq 1$.

**Definition 5.1.1.** For $k \in \{0, 1, \ldots, h\}$, the $k$-th absolute higher Witt index $j_k(\varphi)$ of $\varphi$ is defined to be $i_0(\varphi_{F_k})$. For $k \in \{1, 2, \ldots, h\}$, the $k$-th relative higher Witt index $i_k(\varphi)$ of $\varphi$ is defined to be the difference

$$i_k(\varphi) := j_k(\varphi) - j_{k-1}(\varphi).$$

The $0$-th relative higher Witt index of $\varphi$ is the usual Witt index $i_0(\varphi)$.

It follows from the definition that

$$0 \leq j_0(\varphi) < j_1(\varphi) < \cdots < j_h(\varphi) = \lfloor(\dim \varphi)/2\rfloor.$$
Moreover, it can be shown that the set \( \{ j_0(\varphi), \ldots, j_h(\varphi) \} \) of absolute higher Witt indices of \( \varphi \) is equal to the set of all Witt indices \( i_0(\varphi_K) \) for \( K \) an extension field of \( F \).

### 5.2 The Witt index of \( \varphi_{F(X_2)} \)

We now prove the two results on higher Witt indices for later use.

**Proposition 5.2.1.** If \( \deg \text{CH}(X_2) = 4\mathbb{Z} \), then \( j_1(\varphi) = 1 \).

**Proof.** Let \( K \) be a field of degree 2 over \( F \) such that the anisotropic part of \( \varphi \) has a rational point over \( K \). Then \( i_0(\varphi_K) > i_0(\varphi) \). By [10, Prop. 25.1], it follows that \( i_0(\varphi_K) \geq j_1(\varphi) \). If \( j_1(\varphi) \geq 2 \) then so is \( i_0(\varphi_K) \), which implies that the variety \( X_2 \) has a rational point over \( K \). Since \( K \) has degree 2 over \( F \), this contradicts the assumption. Thus \( j_1(\varphi) = 1 \) and \( j_0(\varphi) = 0 \). \( \square \)

From this proposition, we see that the hypothesis of our Theorem 4.2.4 implies

\[
j_2(\varphi) = j_1(\varphi) + i_2(\varphi) = 1 + 1 = 2.
\]

**Proposition 5.2.2.** If \( j_2(\varphi) = 2 \), then \( i_0(\varphi_{F(X_2)}) = 2 \).

**Proof.** The variety \( X_2 \) has a rational point over \( F(X_2) \), so \( i_0(\varphi_{F(X_2)}) \geq 2 \).

We prove that \( i_0(\varphi_{F(X_2)}) \leq 2 \) by contradiction. If \( \tilde{\varphi} \) is the anisotropic part of \( \varphi_{F(\varphi)} \), then \( \varphi_{F(\varphi)} \simeq \mathbb{H} \perp \tilde{\varphi} \), since our assumption \( j_2(\varphi) = 2 \) implies that \( j_1(\varphi) = 1 \). We define two varieties over \( F' := F(\varphi) \). Let \( Y_1 \) be the projective quadric corresponding to \( \tilde{\varphi} \), and let \( Y_2 \) be the variety of totally isotropic subspaces of dimension 2 with respect to \( \tilde{\varphi} \). Since

\[
i_0(\varphi_{F'(Y_1)}) = i_0(\varphi_{F(\varphi)(\tilde{\varphi})}) = j_2(\varphi) = 2,
\]

26
the variety $X_2$ has a rational point over $F'(Y_1)$. If $i_0(\varphi_{F(X_2)}) \geq 3$, then it follows that $i_0(\tilde{\varphi}_{F'((X_2)_{F'})}) \geq 2$, so $Y_2$ has a rational point over $F'((X_2)_{F'})$. We thus have two rational morphisms between varieties over $F'$:

$$Y_1 \dashrightarrow (X_2)_{F'} \dashrightarrow Y_2.$$ 

By [4, Lem. 6.1], since $X_2$ is smooth and $Y_2$ is complete, there exists a rational morphism from $Y_1$ to $Y_2$, i.e. $(Y_2)_{F'(Y_1)}$ has a rational point. But this is a contradiction, since $j_1(\tilde{\varphi}) = j_2(\varphi) - 1 = 1$. 

\qed
CHAPTER 6

Motivic decomposition

Chernousov and Merkurjev have found a motivic decomposition, given in terms of the root system of a semisimple algebraic group $G$, for the product $X \times X'$ of two projective $G$-homogeneous varieties [6].

After defining the category $\text{Corr}(F)$ of graded correspondences, with objects called motives, we go on to describe the motivic decomposition for groups of inner type. In the final two sections, we compute the decomposition for the varieties $X_e(A)$ with $e = (\text{ind } A)/2$ a power of 2 and for the varieties $X_2(\varphi)$ with $\varphi$ a nondegenerate quadratic form on a vector space of odd dimension.

6.1 Definition

Let $X$ and $Y$ be smooth complete schemes over a field $F$, and let $X_1, X_2, \ldots, X_n$ be the irreducible components of $X$ of dimensions $d_1, d_2, \ldots, d_n$, respectively.

Definition 6.1.1. For any integer $i$, a correspondence $\alpha$ between $X$ and $Y$ of degree $i$ is an element of the direct sum of Chow groups

$$\text{Corr}_i(X,Y) := \bigoplus \text{CH}_{i+k}(X_k \times Y),$$

written $\alpha : X \rightsquigarrow Y$.

We define a category $\mathbf{C}(F)$ with objects of the form $(X,n)$, where $X$ is a smooth complete scheme over $F$ and $n \in \mathbb{Z}$. A morphism between two objects
\((X, n)\) and \((Y, m)\) of \(\mathbf{C}(F)\) is an element of \(\text{Corr}_{n-m}(X, Y)\). Composition of morphisms and the push-forward along a morphism are defined as in [10, Chapter XII].

**Definition 6.1.2.** The category \(\text{Corr}(F)\) of graded correspondences over a field \(F\) is the additive completion of the preadditive category \(\mathbf{C}(F)\) [10, §93]. The objects of the category \(\text{Corr}(F)\) are called (Chow) motives. We will write \(\mathcal{M}(X)(i)\) for the motive \((X, i)\) and \(\mathcal{M}(X)\) for \((X, 0)\).

There is a functor from the category of smooth complete schemes over \(F\) to \(\text{Corr}(F)\) which takes a scheme \(X\) to \(\mathcal{M}(X)\) and a morphism of schemes \(X \to Y\) to the sum of the cycles represented by the closures of the graphs of the morphisms \(X_i \to Y\), where the \(X_i\) are the irreducible components of \(X\) as above.

### 6.2 Motivic decomposition

Given a semisimple algebraic group \(G\), we associate with it the root system data and notation described in chapter 3.

Fix two subsets \(S, S' \subset \Pi\) of simple roots. Denote by \(\Sigma_{S'}\) the root subsystem in \(\Sigma\) generated by \(S'\), and let \(\Sigma_{S'}^+\) be the positive roots of \(\Sigma_{S'}\). Let \(W_S\) and \(W_{S'}\) be the subgroups of the Weyl group \(W\) of the root system \(\Sigma\) which are generated by the sets of reflections with respect to the roots in \(S\) and \(S'\), respectively.

Let \(\Delta\) be the set of all double cosets \(D \in W_S \backslash W/W_{S'}\).

**Theorem 6.2.1** (Chernousov, Merkurjev). For projective homogeneous varieties \(X = H_S\) and \(X' = H_{S'}\) of a semisimple algebraic group \(G\) over \(F\) of inner type, the Chow motives

\[
\mathcal{M}(X \times X') \simeq \bigoplus_{D \in \Delta} \mathcal{M}(H_{RD})(l_D)
\]
are isomorphic, where \( w \) is an element in \( D \) of minimal length \( l_D \) and

\[
R_D := \{ \alpha \in S | w^{-1}(\alpha) \in \Sigma_{S'}^+ \} \subset \Pi.
\]

### 6.3 Generalized Severi-Brauer varieties \( X_\epsilon(A) \)

We once again take \( e = 2^a \) for \( a \in \mathbb{N} \) and a central division algebra \( A \) with \( n := \text{ind} A = 2^{a+1} \). Following the prescription in the previous section, we decompose the Chow motive of \( X_\epsilon(A) \times X_\epsilon(A) \). See also [7] for examples of similar computations.

As noted in chapter 4, \( X_\epsilon(A) \) is a projective \( \text{PGL}(A) \)-homogeneous variety of type \( S = \Pi \setminus \{ \alpha_\epsilon \} \), where

\[
\Pi = \{ \alpha_1 := \epsilon_1 - \epsilon_2, \ldots, \alpha_{n-1} := \epsilon_{n-1} - \epsilon_n \}
\]

is a set of simple roots for the root system \( \Sigma \subset \mathbb{R}^n \) of \( \text{PGL}(A) \).

There are \( e + 1 \) double cosets \( D \in W_S \setminus W/W_S \) with representatives \( w \) as follows, where \( w_{\alpha_k} \) denotes the reflection induced by the root \( \alpha_k \).

\[
1 \quad w_{\alpha_\epsilon} \quad (w_{\alpha_\epsilon} w_{\alpha_{\epsilon-1}})(w_{\alpha_{\epsilon+1}} w_{\alpha_\epsilon}) \quad (w_{\alpha_\epsilon} w_{\alpha_{\epsilon-1}} w_{\alpha_{\epsilon-2}})(w_{\alpha_{\epsilon+1}} w_{\alpha_\epsilon} w_{\alpha_{\epsilon-1}})(w_{\alpha_{\epsilon+2}} w_{\alpha_{\epsilon+1}} w_{\alpha_\epsilon}) \quad \vdots \quad (w_{\alpha_\epsilon} \cdots w_{\alpha_1}) \cdots (w_{\alpha_{2e-1}} \cdots w_{\alpha_\epsilon})
\]

The subset \( R_D \subset \Pi \) associated to \( w = 1 \) is of course \( \Pi \setminus \{ \alpha_\epsilon \} \). The general nontrivial representative

\[
w = w^{-1} = (w_{\alpha_\epsilon} \cdots w_{\alpha_{\epsilon-i}}) \cdots (w_{\alpha_{e+i}} \cdots w_{\alpha_\epsilon}),
\]
for $i \in \{0, \ldots, e - 1\}$, has the effect on $\mathbb{R}^n$ of switching the tuple of standard basis vectors $(\varepsilon_{e-i}, \ldots, \varepsilon_e)$ with the tuple $(\varepsilon_{e+1}, \ldots, \varepsilon_{e+i})$. The resulting $R_D$ is therefore

$$\Pi \setminus \{\alpha_{e-(i+1)}, \alpha_e, \alpha_{e+i+1}\}$$

for $i = 0, \ldots, e - 2$, and $\Pi \setminus \{\alpha_e\}$ for $i = e - 1$.

From Theorem 6.2.1, we deduce the following decomposition of the Chow motive of $X_e \times X_e$, where the relation between the indices $i$ above and $l$ below is $l = i + 1$:

$$\mathcal{M}(X_e \times X_e) \simeq \mathcal{M}(X_e) \oplus \bigoplus_{l=1}^{e-1} \mathcal{M}(X_{e-l,e+l})(l^2) \oplus \mathcal{M}(X_e)(e^2). \quad (6.1)$$

### 6.4 Orthogonal Grassmannians $X_2(\varphi)$

Let $\varphi$ be a nondegenerate quadratic form on a vector space $V$ of dimension $2n + 1$ over a field $F$. We decompose the motive of $X_2(\varphi) \times X_2(\varphi)$.

We recall from chapter 4 that $X_2$ is a projective $O^+(\varphi)$-homogeneous variety of type $S = \Pi \setminus \{\alpha_2\}$, where

$$\Pi = \{\alpha_1 := \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{n-1} := \varepsilon_{n-1} - \varepsilon_n, \alpha_n := \varepsilon_n\}$$

is a set of simple roots for the root system $\Sigma \subset \mathbb{R}^n$ of $O^+(\varphi)$.

When $n \geq 4$, there are six double cosets $D \in W_S \backslash W/W_S$ with representatives $w$ listed in the first column below (where we write $\alpha_k$ when we mean the reflection $w_{\alpha_k}$). The second column lists the effect of $w^{-1}$ on the first four $\varepsilon_i$ (the rest not being affected). The third column gives the subset $R_D \subset \Pi$ associated to $w$.

When $n = 3$, there are only five double cosets. In this case, the table may be amended by deleting the final row and removing all mention of $\varepsilon_4$ from the
remaining rows.

<table>
<thead>
<tr>
<th>$w$ ($\alpha_k$ here means $w_{\alpha_k}$)</th>
<th>$w^{-1}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$</th>
<th>$R_D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$</td>
<td>$\Pi{\alpha_2}$</td>
</tr>
<tr>
<td>$\alpha_2 \cdots \alpha_n \cdots \alpha_2$</td>
<td>$(\varepsilon_1, -\varepsilon_2, \varepsilon_3, \varepsilon_4)$</td>
<td>$\Pi{\alpha_1, \alpha_2}$</td>
</tr>
<tr>
<td>$(\alpha_2 \cdots \alpha_n \cdots \alpha_2)\alpha_1(\alpha_2 \cdots \alpha_n \cdots \alpha_2)$</td>
<td>$(-\varepsilon_2, -\varepsilon_1, \varepsilon_3, \varepsilon_4)$</td>
<td>$\Pi{\alpha_2}$</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>$(\varepsilon_1, \varepsilon_3, \varepsilon_2, \varepsilon_4)$</td>
<td>$\Pi{\alpha_1, \alpha_2, \alpha_3}$</td>
</tr>
<tr>
<td>$\alpha_2\alpha_1(\alpha_3 \cdots \alpha_n \cdots \alpha_2)$</td>
<td>$(\varepsilon_3, -\varepsilon_2, \varepsilon_1, \varepsilon_4)$</td>
<td>$\Pi{\alpha_1, \alpha_2, \alpha_3}$</td>
</tr>
<tr>
<td>$(\alpha_2\alpha_1)(\alpha_3\alpha_2)$</td>
<td>$(\varepsilon_3, \varepsilon_4, \varepsilon_1, \varepsilon_2)$</td>
<td>$\Pi{\alpha_2, \alpha_4}$</td>
</tr>
</tbody>
</table>

From Theorem 6.2.1, we deduce the following decomposition of the motive of $X_2 \times X_2$, where the last summand is removed for the case $n = 3$:

$$
\mathcal{M}(X_2 \times X_2) \simeq \mathcal{M}(X_2) \oplus \mathcal{M}(X_{1,2})(2n - 3) \oplus \mathcal{M}(X_2)(4n - 5) \\
\oplus \mathcal{M}(X_{1,2,3})(1) \oplus \mathcal{M}(X_{1,2,3})(2n - 2) \oplus [\mathcal{M}(X_{2,4})] .
$$

(6.2)
CHAPTER 7

Correspondences and \( p \)-incompressibility

To prove our main results in chapter 9, we will use a criterion for 2-incompressibility of a smooth complete variety \( X \) involving the multiplicities of a correspondence \( \delta \in \text{CH}_{\dim X}(X \times X) \) and its transpose. In the present chapter, we first prove the criterion, and then find conditions under which it holds for \( \delta \) in certain direct summands of a decomposition of \( \text{CH}_{\dim X}(X \times X) \).

7.1 Criterion for \( p \)-incompressibility

We briefly recall some terminology from [10, §62 and §75]. Let \( X \) and \( Y \) be schemes with \( X \) equidimensional. Then a correspondence of degree zero \( \delta : X \rightsquigarrow Y \) is just a cycle \( \delta \in \text{CH}_{\dim X}(X \times Y) \). The multiplicity \( \text{mult}(\delta) \) of such a \( \delta \) is the integer satisfying \( \text{mult}(\delta) \cdot [X] = p_*(\delta) \), where \( p_* \) is the push-forward homomorphism

\[
p_* : \text{CH}_{\dim X}(X \times Y) \to \text{CH}_{\dim X}(X) = \mathbb{Z} \cdot [X].
\]

The exchange isomorphism \( X \times Y \to Y \times X \) induces an isomorphism

\[
\text{CH}_{\dim X}(X \times Y) \to \text{CH}_{\dim X}(Y \times X)
\]

sending a cycle \( \delta \) to its transpose \( \delta^t \).

Proposition 7.1.1. A smooth complete variety \( X \) is \( p \)-incompressible if for any
correspondence \( \delta : X \rightrightarrows X \) of degree zero, \( \text{mult}(\delta) \) is divisible by \( p \) if and only if \( \text{mult}(\delta^t) \) is divisible by \( p \).

**Proof.** Assume the statement on multiplicities holds for a given \( X \), and suppose we have \( f : X' \to X \) and a dominant \( g : X' \to X \) with \( F(X')/F(X) \) finite of degree prime to \( p \). Let \( \delta \in \text{CH}(X \times X) \) be the push-forward of the class \([X']\) along the induced morphism \((g, f) : X' \to X \times X\). By assumption, \( \text{mult}(\delta) \) is prime to \( p \), and thus \( \text{mult}(\delta^t) \) is also prime to \( p \). It follows that \( f_*([X']) \) is a prime to \( p \) multiple of \([X]\) and in particular is nonzero, so \( f \) is dominant. \( \square \)

### 7.2 Even multiplicities

In order to prove 2-incompressibility of a smooth complete variety \( X \), it is therefore enough to check that the multiplicity of a correspondence \( \delta : X \rightrightarrows X \) of degree zero is congruent (modulo 2) to that of its transpose. In this section, we deduce such a result from the condition \( \deg \text{CH}_0(Y_{F(X)}) \subset 2\mathbb{Z} \) for correspondences which “come from” cycles on a variety \( Y \). This condition is equivalent to \( Y \) having no rational point over any odd-degree extension of the function field of \( X \).

**Proposition 7.2.1.** Let \( X \) and \( Y \) be smooth complete varieties. Suppose that \( \deg \text{CH}_0(Y_{F(X)}) \subset 2\mathbb{Z} \), and let the correspondence \( \alpha : Y \rightrightarrows X \times X \) induce an embedding

\[
\alpha_* : \text{CH}_r(Y) \hookrightarrow \text{CH}_{\dim X}(X \times X)
\]

for some \( 0 \leq r \leq \dim Y \). Then for any \( \delta \) in the image of \( \alpha_* \),

\[
\text{mult}(\delta) \equiv 0 \equiv \text{mult}(\delta^t) \pmod{2}.
\]
Proof. Consider the diagram below of fiber products, where we select either of the projections $p_i$ and choose the other morphisms accordingly.

\[
\begin{array}{c}
Y_{F(X)}
\end{array}
\]

\[
\begin{array}{c}
(Y \times X)_{F(X)} \rightarrow X_{F(X)} \rightarrow \text{Spec } F(X)
\end{array}
\]

\[
\begin{array}{c}
Y \times X \times X \rightarrow X \times X \xrightarrow{p_1 \quad p_2} X
\end{array}
\]

\[
\begin{array}{c}
Y \times X \times X \rightarrow Y
\end{array}
\]

Taking push-forwards and pull-backs, we get the following diagram which commutes except for the triangle at the bottom. The push-forward by $p_i$ takes a cycle $\delta \in \text{CH}_{\text{dim} \ X}(X \times X)$ to $\text{mult}(\delta)$ if we chose the first projection $p_1$ or to $\text{mult}(\delta^t)$ if we chose the second projection $p_2$.

\[
\begin{array}{c}
\text{CH}_0(Y_{F(X)})
\end{array}
\]

\[
\begin{array}{c}
\text{CH}_0((Y \times X)_{F(X)}) \rightarrow \text{CH}_0(X_{F(X)}) \rightarrow \mathbb{Z}
\end{array}
\]

\[
\begin{array}{c}
\text{CH}_{\text{dim} \ X}(Y \times X \times X) \rightarrow \text{CH}_{\text{dim} \ X}(X \times X) \rightarrow \mathbb{Z}
\end{array}
\]

\[
\begin{array}{c}
\text{CH}_r(Y)
\end{array}
\]

Any $\delta \in \text{im}(\alpha_*)$ also lies in the image of $\text{CH}_{\text{dim} \ X}(Y \times X \times X)$, by the definition of the push-forward. Chasing through the diagram, one sees that $\text{mult}(\delta)$ (and similarly $\text{mult}(\delta^t)$) must lie in $\text{deg } \text{CH}_0(Y_{F(X)})$, which is contained in $2\mathbb{Z}$ by assumption. \qed
7.3 Applications via indices

We now detail the two cases in which we want to apply Proposition 7.2.1. We keep track of the index of a central simple algebra $A$ and the Witt index of a quadratic form $\varphi$ to check the degree hypothesis of 7.2.1 for certain projective homogeneous varieties $Y$.

**Proposition 7.3.1.** Let $A$ be a central division algebra with $\text{ind} A = 2^e$ a power of 2. Let $F \ell := X_{d_1,d_2,\ldots,d_k}(A)$ with $d := \gcd(e, d_1, d_2, \ldots, d_k) < e$, and let the correspondence $\alpha : F \ell \rightsquigarrow X_e \times X_e$ induce an embedding

$$\alpha_* : \text{CH}_r(F \ell) \hookrightarrow \text{CH}_{e^2}(X_e \times X_e).$$

Then for any $\delta$ in the image of $\alpha_*$, $\text{mult}(\delta) \equiv 0 \equiv \text{mult}(\delta^t) \pmod{2}$.

**Proof.** By Proposition 7.2.1, it is enough to show that $\text{deg} \, \text{CH}_0 \left( (F \ell)_{F(X_e)} \right) \subset 2\mathbb{Z}$. Note that

$$F \ell_{F(X_e)} = X_{d_1,d_2,\ldots,d_k}(A)_{F(X_e)} \simeq X_{d_1,d_2,\ldots,d_k}(A_{F(X_e)}),$$

where $A_{F(X_e)}$ has index equal to $\gcd(2e, e) = e$ [24, Th. 2.5]. If some element of $\text{CH}_0 \left( (F \ell)_{F(X_e)} \right)$ has odd degree, then there exists a field extension $K/F(X_e)$ of odd degree over which $(F \ell)_{F(X_e)}$ has a rational point. By Proposition 4.2.1, $X_d(A_{F(X_e)})$ also has a rational point over $K$. Thus $\text{ind} A_K$ divides $d < e$, which contradicts $\text{ind} A_{F(X_e)} = e$, since an odd degree extension cannot reduce the index of $A_{F(X_e)}$ [23, Th. 3.15a].

**Proposition 7.3.2.** Let $\varphi$ be a nondegenerate quadratic form over $F$ and assume $i_0(\varphi_{F(X_2)}) = 2$. Let $F \ell := X_{d_1,d_2,\ldots,d_s}(\varphi)$ with $d_s > 2$, and let the correspondence $\alpha : F \ell \rightsquigarrow X_2 \times X_2$ induce an embedding

$$\alpha_* : \text{CH}_r(F \ell) \hookrightarrow \text{CH}_{4n-5}(X_2 \times X_2).$$

Then for any $\delta$ in the image of $\alpha_*$, $\text{mult}(\delta) \equiv 0 \equiv \text{mult}(\delta^t) \pmod{2}$. 

36
Proof. By Proposition 7.2.1, it is enough to show that $\deg \text{CH}_0 \left( \left( Fl \right)_{F(X_2)} \right) \subset 2\mathbb{Z}$.

By the assumption on the Witt index, $\varphi_{F(X_2)} \simeq 2\mathbb{H} \perp \psi$ for some anisotropic quadratic form $\psi$ over $F(X_2)$. In order for the variety $Fl_{F(X_2)}$ to have a rational point over a field extension $K$ of $F(X_2)$, $\psi$ must be isotropic over $K$, due to the assumption $d_s > 2$. By Springer’s Theorem, if the degree of the extension $K$ is finite then it must be divisible by 2, so

$$\deg \text{CH}_0 \left( \left( Fl \right)_{F(X_2)} \right) \subset 2\mathbb{Z}.$$
CHAPTER 8

Combinatorial shapes

Given a quadratic form $\varphi$ over $F$, we fix an extension field $E/F$ such that $\varphi_E$ is split and define $\bar{\varphi} := \varphi_E$ and $\bar{X}_d := (X_d)_E$. Let $\overline{\text{CH}}(X_d)$, called the reduced Chow group, denote the image of the change of field homomorphism $\text{CH}(X_d) \to \text{CH}(\bar{X}_d)$. Elements of $\overline{\text{CH}}(X_d)$ will be called rational cycles.

In this chapter we prove a technical lemma, based on a characterization given by Pragacz and Ratajski in [21] of the Chow ring of the variety $\bar{X}_2$. The lemma will be used in the next chapter to check the degree condition of Proposition 7.2.1 for the second direct summand in (6.2).

**Lemma 8.0.3.** If $\gamma \in \text{CH}^r(\bar{X}_2)$ with $r \in \{2n - 3, 2n - 2\}$, then $2\gamma$ is a rational cycle.

8.1 Notation and definitions

We begin by fixing notation and recalling some definitions. Let $X_B := \bar{X}_2$, the variety of 2-dimensional isotropic subspaces of $V$ with respect to the split nondegenerate quadratic form $\bar{\varphi}$. Recall that $V$ has dimension $2n + 1$. Let $X_C$ denote the variety of 2-dimensional isotropic subspaces of a vector space $W$ of dimension $2n$ with respect to a nondegenerate alternating form $\psi$ on $W$.

The next group of definitions are adapted for our purposes from Macdonald's
Definition 8.1.1. A partition is a finite, strictly decreasing sequence of positive integers

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_s^*).$$

The * in our notation will take on the value t for “top” or b for “bottom,” depending on the role the partition plays in a shape, defined below. The length \(l(\lambda^*)\) of the partition above is just \(s\), while the weight of the partition is defined to be \(|\lambda^*| := \sum_{k=1}^{s} \lambda_k^*\). The empty partition, denoted \(\emptyset\), is the sequence with no terms.

Partitions are visualized as diagrams of boxes. The diagram \(D^*_\lambda\) of a partition \(\lambda^*\) has \(\lambda_k^*\) boxes in the \(k\)th row, beginning with the top row. For example, the partition \((4, 3, 1)\) has diagram:

\[
\begin{array}{cccc}
\downarrow & 
\downarrow & 
\downarrow & 
\downarrow \\
\Box & 
\Box & 
\Box & 
\Box \\
\Box & 
\Box & 
\Box & 
\Box \\
\Box & 
\Box & 
\Box & 
\Box \\
\end{array}
\]

Note that the length of a partition is just the number of rows in its diagram, while the weight is just the number of boxes.

A skew diagram \(D^*_\mu \setminus D^*_\lambda\) is obtained by removing the boxes in the intersection of \(D^*_\mu\) and \(D^*_\lambda\) from \(D^*_\mu\). For example, the skew diagram \((4, 3, 1) \setminus (2, 1)\) is as follows:

\[
\begin{array}{cccc}
\downarrow & 
\downarrow & 
\downarrow & 
\downarrow \\
\Box & 
\Box & 
\Box & 
\Box \\
\Box & 
\Box & 
\Box & 
\Box \\
\Box & 
\Box & 
\Box & 
\Box \\
\end{array}
\]
The remaining definitions are adapted from [21].

**Definition 8.1.2.** With \( n \) fixed as above, a pair \( \lambda = (\lambda^t//\lambda^b) \) of partitions \( \lambda^t \) and \( \lambda^b \) is called a *shape* if \( \lambda^t_1, \lambda^b_1 \leq n, l(\lambda^t) \leq n - 2, l(\lambda^b) \leq 2, \) and \( \lambda^t_{n-2} > l(\lambda^b) \).

The *diagram* \( D_\lambda \) of a shape is drawn by stacking the diagram \( D^t_\lambda \) of the first partition on top of the diagram \( D^b_\lambda \) of the second, with a horizontal line in between. For example, the shape \( ((4,2)//(3)) \) has diagram:

\[
\begin{array}{cccc}
\hline
& & & \\
& & & \\
\hline
& & & \\
& & & \\
\hline
\end{array}
\]

The inequalities in the definition of a shape \( \lambda \) amount to imposing three requirements on its diagram \( D_\lambda \):

1. \( D^t_\lambda \) must fit into a rectangle with height \( n - 2 \) and width \( n \)
2. \( D^b_\lambda \) must fit into a rectangle with height \( 2 \) and width \( n \)
3. \( D_\lambda \) must contain the “triangle” of boxes lying on and above the diagonal beginning at the box in the diagram’s lower-left corner and proceeding to the “northeast”.

We display once again the diagram of the shape \( ((4,2)//(3)) \), this time shading in the triangle of boxes required by the third condition above:
The weight of a shape $\lambda = (\lambda^t//\lambda^b)$ is defined in terms of the weights of the partitions $\lambda^t, \lambda^b$ as

$$|\lambda| := |\lambda^t| + |\lambda^b| - \binom{n-1}{2}.$$  

This definition is chosen so that the shape $\pi_0 := ((n-2, n-3, \ldots, 1)//\emptyset)$ of minimal weight will have weight 0.

We refer to [21] for the definitions of extremal and related components, $(\mu - \lambda)$-boxes, and compatible shapes.

### 8.2 Multiplication in the Chow rings

The set of all shapes, denoted $\mathcal{P}_2$, can be mapped bijectively onto bases for each of $\text{CH}(X_B)$ and $\text{CH}(X_C)$. We name the maps

$$\sigma : \mathcal{P}_2 \longrightarrow \text{CH}(X_B)$$  

$$\tau : \mathcal{P}_2 \longrightarrow \text{CH}(X_C)$$

and we call cycles in the images of the maps basic cycles.

For $i = 0, 1, 2$, consider the shapes $\pi_i := ((n - 2 + i, n - 3, n - 4, \ldots, 1)//\emptyset)$. When $n = 5$, their diagrams are as follows:

![Diagrams](image)

If we set $\sigma_i := \sigma(\pi_i)$, $\tau_i := \tau(\pi_i)$, then $\sigma_0, \tau_0$ are the respective multiplicative identities of the Chow rings, while $\sigma_1, \sigma_2$ (resp. $\tau_1, \tau_2$), called special cycles, are the nontrivial Chern classes of the tautological bundle over $X_B$ (resp. $X_C$).
Pragacz and Ratajski prove that $\tau_1, \tau_2$ algebraically generate $\text{CH}(X_C)$ (Cor. 1.8 and Lem. 3.2), while $\sigma_1, \sigma_2$ only generate $\text{CH}(X_B)$ after tensoring with $\mathbb{Z}[1/2]$ (Thm. 10.1).

With the weight $|\lambda|$ of a shape defined as above, we have $\sigma(\lambda) \in \text{CH}^{|\lambda|}(X_B)$ and $\tau(\lambda) \in \text{CH}^{|\lambda|}(X_C)$. In particular, $\text{codim}(\sigma_i) = \pi_i = i$ and $\text{codim}(\tau_i) = |\pi_i| = i$.

The multiplication rules in $\text{CH}(X_B)$ and $\text{CH}(X_C)$ are very similar, differing only by some factors of 2 in certain multiplicities. Indeed, for any shape $\lambda \in \mathcal{P}_2$, $i = 1, 2$, we have the Pieri-type formulas

$$\sigma(\lambda) \cdot \sigma_i = \sum 2^{e_B(\lambda, \mu)} \sigma(\mu)$$

$$\tau(\lambda) \cdot \tau_i = \sum 2^{e_C(\lambda, \mu)} \tau(\mu)$$

for multiplying a basic cycle by a special cycle [21, Thms. 2.2 and 10.1]. Here, the sums are over all $\mu$ compatible with $\lambda$ satisfying $|\mu| = |\lambda| + i$, and $e_B(\lambda, \mu), e_C(\lambda, \mu)$ are the cardinalities of certain sets of components of the skew diagram $D^b_\mu \setminus D^b_\lambda$. For our purposes, what we need is that for compatible $\lambda, \mu$ with $|\mu| = |\lambda| + i$, the difference $e_B(\lambda, \mu) - e_C(\lambda, \mu)$ equals the number of extremal components of $D^b_\mu \setminus D^b_\lambda$. (This follows from the fact that an extremal component is not related and has no $(\mu - \lambda)$-boxes lying over it, by parts 2 and 4 of the definition [21, Def. 2.1] of compatible shapes.) The skew diagram $D^b_\mu \setminus D^b_\lambda$ clearly has at most one extremal component. There is exactly one extremal component if and only if $l(\mu^b) > l(\lambda^b)$, which by part 5 of the definition of compatible shapes is equivalent to $l(\mu^b) = l(\lambda^b) + 1$. There are no extremal components if and only if $l(\mu^b) = l(\lambda^b)$.

Putting all of this together, we conclude that

$$e_B(\lambda, \mu) - e_C(\lambda, \mu) = l(\mu^b) - l(\lambda^b) \in \{0, 1\}. \quad (8.1)$$

To describe the product of several special cycles, we need to extend the notion
of compatibility to sequences of shapes. For nonnegative integers $a_1, a_2$, define a compatible $(a_1, a_2)$-chain to be a sequence of shapes

$$
\Lambda = (\pi_0 = \lambda_0, \lambda_1, \ldots, \lambda_{a_1+a_2})
$$

such that for $i = 1, 2, \ldots, (a_1+a_2)$, the shapes $\lambda_i$ and $\lambda_{i-1}$ are compatible, $|\lambda_i| - |\lambda_{i-1}| \in \{1, 2\}$, and $|\lambda_{a_1+a_2}| = a_1 + 2a_2$.

We now can write down formulas for an arbitrary product of special cycles in $\text{CH}(X_B)$ or $\text{CH}(X_C)$. In $\text{CH}(X_B)$, the formula is

$$
\sigma_1^{a_1} \cdot \sigma_2^{a_2} = \sum_{\text{compatible } (a_1, a_2)-\text{chains } \Lambda=(\lambda_0, \lambda_1, \ldots, \lambda_{a_1+a_2})} 2^{b_\Lambda} \sigma(\lambda_{a_1+a_2}),
$$

where

$$
b_\Lambda = e_B(\lambda_0, \lambda_1) + e_B(\lambda_1, \lambda_2) + \cdots + e_B(\lambda_{a_1+a_2-1}, \lambda_{a_1+a_2}),
$$

and in $\text{CH}(X_C)$, the formula is

$$
\tau_1^{a_1} \cdot \tau_2^{a_2} = \sum_{\text{compatible } (a_1, a_2)-\text{chains } \Lambda=(\lambda_0, \lambda_1, \ldots, \lambda_{a_1+a_2})} 2^{c_\Lambda} \tau(\lambda_{a_1+a_2}),
$$

where

$$
c_\Lambda = e_C(\lambda_0, \lambda_1) + e_C(\lambda_1, \lambda_2) + \cdots + e_C(\lambda_{a_1+a_2-1}, \lambda_{a_1+a_2}).
$$

It follows from equation (8.1) that

$$
b_\Lambda - c_\Lambda = [l(\lambda_b^0) - l(\lambda_b^0)] + [l(\lambda_b^1) - l(\lambda_b^0)] + \cdots + [l(\lambda_b^{a_1+a_2}) - l(\lambda_b^{a_1+a_2-1})]
$$

$$
= l(\lambda_b^{a_1+a_2}) - l(\emptyset)
$$

$$
= l(\lambda_b^{a_1+a_2}). \quad (8.2)
$$

### 8.3 Proof of the lemma

We now are ready to prove Lemma 8.0.3.
Proof. It is enough to take \( \gamma \) to be a basic cycle, say \( \gamma = \sigma(\lambda) \) for some shape \( \lambda \) of weight \( r \). Since \( \tau_1 \) and \( \tau_2 \) generate the ring \( \text{CH}(X_C) \), there exist integers \( u_j \) such that

\[
\tau(\lambda) = \sum_{j=0}^{\lfloor r/2 \rfloor} u_j \left( \tau_1^{r-2j} \cdot \tau_2^j \right) \in \text{CH}^r(X_C).
\]

Define

\[
\gamma' := \sum_{j=0}^{\lfloor r/2 \rfloor} u_j \left( \sigma_1^{r-2j} \cdot \sigma_2^j \right) \in \text{CH}^r(X_B).
\]

This is a rational cycle, since \( \sigma_1, \sigma_2 \) are Chern classes of the tautological bundle over \( X_B = \tilde{X}_2 \) and are therefore defined over the base field \( F \).

It remains to show that \( \gamma' = 2\gamma \). The key is that for any shape \( \lambda = (\lambda_t/\lambda_b) \) of weight \( 2n - 3 \) or \( 2n - 2 \), \( l(\lambda_b) = 1 \). Indeed, it follows easily from the conditions imposed in the definition of a shape \( \lambda \) that \( l(\lambda_b) = 0 \) implies \( |\lambda| \leq 2n - 4 \), and \( l(\lambda_b) = 2 \) (the greatest value possible) implies \( |\lambda| \geq 2n - 1 \). The case \( n = 5 \) is illustrated below, where we shade the “\( \pi_0 \) boxes” which don’t contribute to the weight \( |\lambda| \). Diagram I corresponds to the shape of maximal weight \( 2n - 4 = 6 \) among shapes \( \lambda \) with \( l(\lambda_b) = 0 \). Diagram II corresponds to the shape of minimal weight \( 2n - 1 = 9 \) among shapes \( \lambda \) with \( l(\lambda_b) = 2 \).

Expanding the products in the expression for \( \tau(\lambda) \), we get

\[
\tau(\lambda) = \sum_{j=0}^{\lfloor r/2 \rfloor} u_j \sum_{\text{compatible } (r-2j,j)-\text{chains } \Lambda=(\lambda_0,\lambda_1,...,\lambda_{r-j})} 2^{\epsilon(\Lambda)} \tau(\lambda_{r-j}).
\]
The maps $\sigma$ and $\tau$ induce a group isomorphism \( \sigma \circ \tau^{-1} : \text{CH}(X_C) \to \text{CH}(X_B) \) which when applied to the equation above yields

\[
\sigma(\lambda) = \sum_{j=0}^{\lfloor r/2 \rfloor} u_j \sum_{\text{compatible } (r-2j, j)-\text{chains} \Lambda=(\lambda_0, \lambda_1, \ldots, \lambda_{r-j})} 2^{c_\Lambda} \sigma(\lambda_{r-j}). \tag{8.3}
\]

On the other hand, expanding the expression for $\gamma'$ yields

\[
\gamma' = \sum_{j=0}^{\lfloor r/2 \rfloor} u_j \sum_{\text{compatible } (r-2j, j)-\text{chains} \Lambda=(\lambda_0, \lambda_1, \ldots, \lambda_{r-j})} 2^{b_\Lambda} \sigma(\lambda_{r-j}), \tag{8.4}
\]

which differs from the expression for $\sigma(\lambda)$ only in that the exponent $c_\Lambda$ has been changed to $b_\Lambda$. By equation (8.2) and our length computation above, $b_\Lambda - c_\Lambda = l(\lambda_{r-j}^b) = 1$ for any compatible $(r-2j, j)$-chain $\Lambda$, since

\[
|\lambda_{r-j}| = (r - 2j) + 2j = r \in \{2n - 3, 2n - 2\}.
\]

Thus each term on the right-hand side of (8.4) is twice the corresponding term on the right-hand side of (8.3) and $\gamma' = 2\sigma(\lambda) = 2\gamma$. \qed
CHAPTER 9

Incompressibility theorems

In this final chapter, we bring together the preceding results to prove 2-incompressibility for two infinite families of projective homogeneous varieties.

9.1 Generalized Severi-Brauer varieties $X_e(A)$

We first consider our class of generalized Severi-Brauer varieties.

**Theorem 9.1.1.** Let $e = 2^a$, $a \in \mathbb{N}$. For a central division algebra $A$ with $\text{ind } A = 2^{a+1}$, the variety $X_e := X_e(A)$ is 2-incompressible. Thus

$$\text{cdim}_2(X_e) = \text{cdim}(X_e) = \dim(X_e) = e(2e - e) = e^2 = 4^a.$$ 

**Proof.** By Proposition 7.1.1, it suffices to show that for any cycle $\delta \in \text{CH}_{e^2}(X_e \times X_e)$,

$$\text{mult}(\delta) \equiv \text{mult}(\delta^t) \pmod{2}. \quad (9.1)$$

The motivic decomposition (6.1) yields a decomposition of the middle-dimensional component of the Chow group of $X_e \times X_e$:

$$\text{CH}_{e^2}(X_e \times X_e) \simeq \text{CH}_{e^2}(X_e) \oplus \bigoplus_{l=1}^{e-1} \text{CH}_{(e-l)(e+l)}(X_{e-l,e,e+l}) \oplus \text{CH}_0(X_e).$$

It is enough to check the congruence (9.1) for $\delta$ in the image of any of these direct summands.
For all of the summands except the first and the last, we may apply Proposition 7.3.1, since \( \gcd(e, e - l, e, e + l) < e \).

The embedding of the first summand \( CH_e^2(X_e) \) is induced by the diagonal morphism \( X_e \to X_e \times X_e \), so for \( \delta \) in the image of this summand, the multiplicities of \( \delta \) and \( \delta^t \) are equal by symmetry.

For the last summand \( CH_0(X_e) \) we need the following fact.

**Proposition 9.1.2.** Any element of \( CH_0(X_e) \) has even degree.

*Proof.* If \( CH_0(X_e) \) has an element of odd degree, then there exists a field extension \( K/F \) of odd degree over which \( X_e \) has a rational point. By [17, Prop. 1.17], \( \text{ind } A_K \) divides \( e \). Since the degree of \( K \) over \( F \) is relatively prime to \( \text{ind } A = 2e = 2^{a+1} \), extension by \( K \) does not reduce the index of \( A \) [23, Th. 3.15a]. Thus \( \text{ind } A = \text{ind } A_K \) divides \( e \), a contradiction. \( \square \)

Let the element \( \gamma \in CH_0(X_e) \) have image \( \delta \in CH_e^2(X_e \times X_e) \). By the proposition, \( \deg(\gamma) \) is even. For some field \( E/F \) over which \( X_e \) has a rational point, we set \( \bar{X}_e := (X_e)_E \). Since \( CH_0(\bar{X}_e) \) is generated by a single element of degree 1, the image of \( \gamma \) in \( CH_0(\bar{X}_e) \) is divisible by 2. It follows that \( \delta \in CH_e^2(\bar{X}_e \times \bar{X}_e) \) is also divisible by 2 and, since multiplicity does not change under field extension, \( \text{mult}(\delta) \) is even. The same argument can be applied to \( \delta^t \), so \( \text{mult}(\delta) \equiv 0 \equiv \text{mult}(\delta^t) \) (mod 2).

We have checked (9.1) for \( \delta \) in the image of each direct summand, hence the proof of the theorem is complete. \( \square \)

### 9.2 Orthogonal Grassmannians \( X_2(\varphi) \)

We finish by considering orthogonal Grassmannians of rank 2.
Theorem 9.2.1. Fix a nondegenerate quadratic form \( \varphi \) on a vector space \( V \) of dimension \( 2n + 1 \), \( n \geq 3 \), over a field \( F \). If \( \deg \text{CH}(X_2) = 4\mathbb{Z} \) and \( i_2(\varphi) = 1 \), then \( X_2 \) is 2-incompressible. In particular,

\[ \text{cdim}_2(X_2) = \text{cdim}(X_2) = \dim(X_2) = 4n - 5. \]

Proof. By Proposition 7.3.2, it suffices to show that for any cycle \( \delta \in \text{CH}_{4n-5}(X_2 \times X_2) \),

\[ \text{mult}(\delta) \equiv \text{mult}(\delta^t) \pmod{2}. \] (9.2)

The motivic decomposition (6.2) yields a decomposition of the middle-dimensional component of the Chow group of \( X_2 \times X_2 \):

\[
\text{CH}_{4n-5}(X_2 \times X_2) \cong \text{CH}_{4n-5}(X_2) \oplus \text{CH}_{2n-2}(X_{1,2}) \oplus \text{CH}_0(X_2) \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \oplus \text{CH}_{4n-6}(X_{1,2,3}) \oplus \text{CH}_{2n-3}(X_{1,2,3}) \oplus \left[ \text{CH}_{4n-9}(X_{2,4}) \right].
\]

It is enough to check the congruence (9.2) for \( \delta \) in the image of any of these direct summands.

The last three summands have \( d_k > 2 \) and so are dealt with by Proposition 7.3.2, the hypothesis of which holds by our Propositions 5.2.1 and 5.2.2.

The embedding of the first summand \( \text{CH}_{4n-5}(X_2) \) is induced by the diagonal morphism \( X_2 \to X_2 \times X_2 \), so the multiplicities are equal by symmetry.

Any element \( \gamma \) of the third summand \( \text{CH}_0(X_2) \) has degree divisible by 4 by assumption, hence its image in the Chow group \( \text{CH}_0(\bar{X}_2) \) is divisible by 4. (Here we use that \( \text{CH}_0(\bar{X}_2) \) is generated by a single element of degree 1.) The image \( \delta \in \text{CH}_{4n-5}(\bar{X}_2 \times \bar{X}_2) \) of \( \gamma \) is then also divisible by 4 and, since multiplicity does not change under field extension, \( \text{mult}(\delta) \equiv 0 \equiv \text{mult}(\delta^t) \pmod{4} \).

The second summand requires our work from chapter 8. Since \( X_{1,2} \) is a
projective bundle over $X_2$, there is a motivic decomposition

$$\mathcal{M}(X_{1,2}) \simeq \mathcal{M}(X_2) \oplus \mathcal{M}(X_2)(1),$$

so that

$$\text{CH}_{2n-2}(X_{1,2}) \simeq \text{CH}_{2n-2}(X_2) \oplus \text{CH}_{2n-3}(X_2).$$

It is enough to consider $\delta$ equal to the image of some $\beta \in \text{CH}_r(X_2)$, where $r \in \{2n - 3, 2n - 2\}$. By the same reasoning as in the previous paragraph, it suffices to show that the image of $\beta$ in $\overline{\text{CH}}_r(X_2) \subset \text{CH}_r(\bar{X}_2)$ is divisible by 2 in $\text{CH}_r(\bar{X}_2)$. Suppose it is not. Then the image $\hat{\beta}$ of $\beta$ in the modulo-2 Chow group

$$\text{Ch}_r(\bar{X}_2) := \text{CH}_r(\bar{X}_2)/2 \text{CH}_r(\bar{X}_2)$$

is nonzero. By [13, Rem. 5.6], the “cellular” variety $\bar{X}_2$ is “2-balanced,” i.e. the bilinear form $⟨\hat{\beta}, \hat{\gamma}\rangle \mapsto \deg(\hat{\beta} \cdot \hat{\gamma})$ on $\text{Ch}(\bar{X}_2)$ is nondegenerate. Hence there exists $\gamma \in \text{CH}'(\bar{X}_2)$ such that

$$\deg(\beta \cdot \gamma) \equiv 1 \pmod{2}.$$

Since $2\gamma$ is rational by our Lemma 8.0.3, we have

$$\deg \overline{\text{CH}}_0(X_2) \ni \deg(\beta \cdot 2\gamma) \equiv 2 \pmod{4}.$$

Degree does not change under field extension, so this contradicts our assumption that $\deg \text{CH}(X_2) = 4\mathbb{Z}$. 

□
References


