One-to-one, onto, invertible

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be any function. The domain of $T$ is $\mathbb{R}^n$ and the codomain of $T$ is $\mathbb{R}^m$. The range of $T$ is the set of all outputs of $T$, namely:

$$\{T(x) \in \mathbb{R}^m | x \in \mathbb{R}^n\}.$$ 

The range is always a subset of the codomain. It may equal the codomain, or it may be smaller than (i.e. “properly contained in”) the codomain.

**Definition.** The function $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if it maps different inputs to different outputs. In symbols, this means

$$x \neq y \implies T(x) \neq T(y),$$

or, equivalently, the contrapositive statement

$$T(x) = T(y) \implies x = y.$$ 

To show that a function is one-to-one, use the second statement. Assume that $T(x) = T(y)$ and use this to show $x = y$.

To show that a function is not one-to-one, it suffices to find two different vectors $x \neq y$ with the same output $T(x) = T(y)$.

**Definition.** The function $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of $T$ equals the codomain $\mathbb{R}^m$.

**Theorem.** For any $T : \mathbb{R}^n \to \mathbb{R}^m$,

$$T \text{ is invertible } \iff T \text{ is one-to-one and } T \text{ is onto.}$$

$T$ one-to-one implies that each element of $\mathbb{R}^m$ is the output for at most one input, while $T$ onto implies that each element of $\mathbb{R}^m$ is the output for at least one input. Together, they imply that each element $w \in \mathbb{R}^m$ is the output for exactly one input, in other words, the equation $T(x) = w$ has exactly one solution for every $w \in \mathbb{R}^m$. The inverse function $T^{-1}$ sends $w$ to this unique solution $x$.

**Theorem.** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. (Note that the dimensions of the domain and codomain are assumed to be the same.) In this case,

$$T \text{ is invertible } \iff T \text{ is one-to-one } \iff T \text{ is onto.}$$

To show that such a linear transformation is invertible, it is enough to show either that it is one-to-one or that it is onto.