7.3 Finding the Eigenvectors of a Matrix

Definition 1 (7.3.1). Let $\lambda$ be an eigenvalue of an $n \times n$ matrix $A$. The $\lambda$-eigenspace of $A$, denoted $E_\lambda$, is defined to be

$$E_\lambda = \ker(A - \lambda I_n)$$

$$= \{v \in \mathbb{R}^n : Av = \lambda v\}$$

$$= \{\lambda\text{-eigenvectors of } A\} \cup \{0\}.$$ 

Note 2. An eigenspace is a subspace, since it is the kernel of the matrix $A - \lambda I_n$. All of the nonzero vectors in $E_\lambda$ are $\lambda$-eigenvectors.

Definition 3 (7.3.2). The dimension of the $\lambda$-eigenspace $E_\lambda = \ker(A - \lambda I_n)$ is called the geometric multiplicity of $\lambda$, written $GM(\lambda)$. We have

$$GM(\lambda) = \dim(E_\lambda)$$

$$= \dim(\ker(A - \lambda I_n))$$

$$= \text{nullity}(A - \lambda I_n)$$

$$= n - \text{rank}(A - \lambda I_n).$$

Definition 4 (7.3.3). Let $A$ be an $n \times n$ matrix. A basis of $\mathbb{R}^n$ consisting of eigenvectors of $A$ is called an eigenbasis for $A$.

Theorem 5 (eigenvectors with distinct eigenvalues are linearly independent).

Let $A$ be a square matrix. If $v_1, v_2, \ldots, v_s$ are eigenvectors of $A$ with distinct eigenvalues, then $v_1, v_2, \ldots, v_s$ are linearly independent.

Note 6. Part (a) of the following theorem is a generalization of the preceding theorem, allowing multiple (linearly independent) eigenvectors with a single eigenvalue.

Theorem 7 (7.3.4, eigenbases and geometric multiplicities).

a) Let $A$ be an $n \times n$ matrix. If we concatenate bases for each eigenspace of $A$, then the resulting eigenvectors $v_1, \ldots, v_s$ will be linearly independent. (Note that $s$ is the sum of the geometric multiplicities of the eigenvalues of $A$.)

b) There exists an eigenbasis for an $n \times n$ matrix $A$ if and only if the sum of the geometric multiplicities of its eigenvalues equals $n$:

$$\sum_{\lambda \text{ of } A} GM(\lambda) = n.$$ 

Theorem 8 (7.3.5, $n$ distinct eigenvalues). If an $n \times n$ matrix has $n$ distinct eigenvalues, then there exists an eigenbasis for $A$. 


Theorem 9 (7.3.6, eigenvalues of similar matrices). Suppose $A$ is similar to $B$. Then

a) $f_A(\lambda) = f_B(\lambda)$. (study only this part for the quiz)

b) $\text{nullity}(A) = \text{nullity}(B)$ and $\text{rank}(A) = \text{rank}(B)$.

c) $A$ and $B$ have the same eigenvalues, with the same algebraic and geometric multiplicities.

d) $\det A = \det B$ and $\text{tr} A = \text{tr} B$.

Note 10. Similar matrices generally do not have the same eigenvectors.

Theorem 11 (7.3.7, algebraic and geometric multiplicity). If $\lambda$ is an eigenvalue of $A$, then

$$GM(\lambda) \leq AM(\lambda).$$

Combining this with earlier results, we get

$$\sum_{\lambda \in \text{eigenvalues of } A} GM(\lambda) \leq \sum_{\lambda \in \text{eigenvalues of } A} AM(\lambda) \leq n.$$